Quantile-based modeling of scale dynamics in financial returns for Value-at-Risk and Expected Shortfall forecasting

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Abstract: We introduce a semiparametric approach for forecasting Value-at-Risk (VaR) and Expected Shortfall (ES) by modeling the conditional scale of financial returns, defined as the difference between two specified quantiles, via restricted quantile regression. Focusing on downside risk, VaR is derived from the left-tail quantile of rescaled returns, and ES is approximated by averaging quantiles below the VaR level. The method delivers robust, distribution-free estimates of extreme losses and captures skewness, heavy tails, and leverage effects. Simulation experiments and empirical analysis show that it often outperforms established models, including GARCH and joint VaR-ES conditional-quantile approaches. An application to daily returns on major international stock indices, spanning the COVID-19 period, highlights its effectiveness in capturing risk dynamics.

JEL classification: C14, C22, G17, G32

Keywords: Conditional scale dynamics, CAViaR models, Multiple quantiles, Robust risk estimation, Financial risk forecasting

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1 Introduction

Forecasting risk measures such as Value-at-Risk (VaR) and Expected Shortfall (ES) is crucial for financial institutions, risk managers, and regulators. VaR, which estimates a conditional quantile in the lower tail of the return distribution, has long been a standard tool for assessing potential losses. Specifically, the VaR for period t at probability level α is defined as

$$\text{VaR}_{r_t}(\alpha) = Q_{r_t}(\alpha \,|\, \mathcal{I}_{t-1}), \tag{1}$$

where r_t denotes the asset return at time t and $Q_{r_t}(\alpha \,|\, \mathcal{I}_{t-1})$ is the αth conditional quantile given the information set $\mathcal{I}_{t-1}$. However, VaR has notable limitations, particularly its inability to account for extreme losses beyond the quantile threshold defined by α, and its potential failure to capture diversification benefits (Embrechts et al., 2002). This is because VaR is not a coherent risk measure; it does not satisfy the property of subadditivity, meaning that the VaR of a combined portfolio can exceed the sum of the VaRs of its individual components. As a result, VaR may inadequately reflect the benefits of diversification.

In contrast, ES is a coherent risk measure, addressing these shortcomings by estimating the expected loss beyond the VaR threshold and, in particular, satisfying subadditivity (Artzner et al., 1999; Acerbi and Tasche, 2002). Conditional on information $\mathcal{I}_{t-1}$, ES for period t at probability level α is defined as

$$\text{ES}_{r_t}(\alpha) = \mathbb{E}[r_t \,|\, r_t \leq \text{VaR}_{r_t}(\alpha), \mathcal{I}_{t-1}], \tag{2}$$

providing the average loss when r_t falls below the VaR threshold in (1). This makes ES a more theoretically sound measure that incentivizes diversification and better accounts for extreme risks. Recognizing these advantages, the Basel Committee, through its Basel III framework and the Fundamental Review of the Trading Book (FRTB), has replaced

VaR with ES as the primary metric for market risk capital requirements (Basel Committee on Banking Supervision, 2019). This regulatory shift underscores the need for robust ES forecasting models that can accurately quantify tail risk under changing market conditions.

ES forecasts can often be derived as a natural extension of many VaR forecasting methods. Nonparametric approaches, including historical simulation and kernel density estimation, generate density forecasts from which both VaR and ES predictions can be obtained. Similarly, parametric methods, typically based on models for the conditional variance, such as GARCH specifications combined with a precise assumption about the conditional return distribution, provide concurrent forecasts for both risk measures.

A prominent semiparametric method for VaR forecasting is quantile regression, specifically the Conditional Autoregressive VaR (CAViaR) models of Engle and Manganelli (2004), which model the conditional quantile directly without relying on specific distributional assumptions. Let $y_t = r_t - \mu_t$, where μ_t represents the conditional location of returns, often assumed to be zero, a constant, or a dynamic process such as AR(1), as in Kuester et al. (2006). A first-order symmetric absolute value (SAV) CAViaR specification is expressed as

$$Q_{y_t}(\alpha \,|\, \mathcal{I}_{t-1}) = \omega(\alpha) + \beta(\alpha) Q_{y_{t-1}}(\alpha \,|\, \mathcal{I}_{t-2}) + \gamma(\alpha)|y_{t-1}|, \tag{3}$$

with the VaR in (1) given by $Q_{r_t}(\alpha \,|\, \mathcal{I}_{t-1}) = \mu_t + Q_{y_t}(\alpha \,|\, \mathcal{I}_{t-1})$. This model treats positive and negative deviations of y_{t-1} symmetrically, relying solely on $|y_{t-1}|$ and thereby implicitly responding to volatility symmetrically. Alternatively, an asymmetric slope (AS) CAViaR specification is given by

$$Q_{y_t}(\alpha \,|\, \mathcal{I}_{t-1}) = \omega(\alpha) + \beta(\alpha) Q_{y_{t-1}}(\alpha \,|\, \mathcal{I}_{t-2}) + \big(\gamma_+(\alpha)\mathbb{1}\{y_{t-1} > 0\} + \gamma_-(\alpha)\mathbb{1}\{y_{t-1} \leq 0\}\big)|y_{t-1}|, \tag{4}$$

where $\mathbb{1}\{\cdot\}$ denotes the indicator function. This formulation allows the model to capture asymmetries in the conditional quantile dynamics by adjusting the slope based on whether the previous centered return was positive or negative, thus making it sensitive to directional changes in past returns. The CAViaR models in (3) and (4) allow the dynamics of $Q_{y_t}(\alpha \,|\, \mathcal{I}_{t-1})$ to vary across different probability levels α and have shown strong performance in empirical studies of VaR forecast accuracy (see, e.g., Şener et al., 2012). However, while CAViaR models excel in predicting VaR, they do not inherently provide a mechanism for generating ES forecasts, as they focus solely on a particular quantile of the return distribution.

Recent advances in risk forecasting have focused on methods that jointly estimate VaR and ES, leveraging the fact that these two risk measures are jointly elicitable (Fissler and Ziegel, 2016). A risk measure is said to be elicitable if it can be uniquely identified as the minimizer of a well-defined expected loss function, also known as a scoring function. In simpler terms, an elicitable risk measure can be evaluated and compared using a loss function that ranks different forecasting methods based on their accuracy. While ES is not elicitable on its own, the joint elicitability of VaR and ES allows for their simultaneous estimation and evaluation using a common loss function. This property has been pivotal in recent developments in financial risk forecasting.

Taylor (2019) proposes a joint VaR-ES model based on the asymmetric Laplace (AL) distribution, where the VaR component is modeled using a CAViaR specification. This approach uses the AL density to construct a likelihood function, whose maximization is equivalent to minimizing a strictly consistent Fissler and Ziegel (2016) scoring function. Similarly, Patton et al. (2019) introduce a semiparametric approach within the Generalized Autoregressive Score (GAS) framework of Creal et al. (2013), which also exploits the Fissler and Ziegel (2016) class of loss functions. Their GAS model dynamically adjusts both VaR and ES estimates based on past observations, specifically reacting to VaR violations while reverting to a long-run mean in the absence of breaches.

In this paper, we introduce a novel semiparametric CAViaR-based framework for forecasting VaR and ES, which we term the Quantile-based Scale Dynamics (QbSD) approach. The QbSD framework extends the widely used GARCH class by replacing the conditional standard deviation with a conditional scale—a measure analogous to the interquartile range (IQR), but more flexible as it is based on the difference between two specified quantiles of the return distribution, similar to Taylor (2005). To model the dynamics of the conditional scale, QbSD employs global CAViaR specifications in which persistence parameters are shared across quantiles to ensure parsimony, while local parameters account for level-specific adjustments. Parameter constraints enforce non-crossing quantile ordering and strictly positive scale values, ensuring internal consistency and robust tail risk estimation.

Beyond its theoretical advantages, the QbSD approach aligns with practical needs in financial risk management. Financial institutions rely on multiple model validations in modern risk management frameworks, particularly for stress-testing and regulatory reporting under Basel III guidelines. Given its flexibility and lack of reliance on specific distributional assumptions, QbSD can serve as a complementary tool for financial institutions in benchmarking VaR and ES estimates against conventional models. This is particularly relevant in periods of financial distress, where traditional parametric models may struggle to adapt to evolving risk dynamics. Furthermore, by providing more accurate tail risk estimates, QbSD can aid institutions in regulatory capital calculations and internal risk monitoring, supporting more resilient decision-making.

To achieve this, QbSD models the dynamics of conditional quantiles, enabling a structured estimation of both VaR and ES. Specifically, VaR is estimated from the left-tail quantile of the rescaled returns, while ES is approximated by averaging quantiles across probability levels below the VaR threshold. This approach leverages the representation of ES as the integral of conditional quantiles across the left tail of the distribution up to the VaR probability level (see, e.g., Acerbi and Tasche, 2002). As a result, the method delivers valid, well-ordered risk estimates across all probability levels, including those in the extreme tails of the distribution.

This approach shares similarities with the Filtered Historical Simulation (FHS) method described by Christoffersen (2012, Section 6.4), which first applies a volatility model—typically a GARCH specification with an assumed innovation distribution—to filter historical returns. FHS then standardizes the returns by dividing them by the conditional volatility estimates, and subsequently computes the quantiles of these residuals to estimate VaR. The average residual loss beyond the VaR estimate is used to calculate ES. In contrast, the semi-parametric QbSD approach does not rely on a specific distributional assumption. Instead, it models the conditional scale dynamics using restricted CAViaR specifications, offering a robust and distribution-free framework for forecasting VaR and ES.

The remainder of the paper is organized as follows. Section 2 develops the QbSD approach for forecasting VaR and ES. Section 3 presents simulation experiments designed to evaluate the performance of the proposed method across a range of market conditions. In Section 4, we assess the QbSD method's effectiveness through an empirical application, using daily returns from major international stock indices, including the volatile period surrounding the COVID-19 pandemic. Finally, Section 5 concludes the paper. Additional numerical and empirical results are provided in the Supplementary material.

2 Quantile-based modeling

2.1 Background

Consider first the unconditional distribution of asset returns and suppose it belongs to the location-scale family, so that

$$r_t = \mu + s\varepsilon_t, \tag{5}$$

where μ is the location parameter, $s > 0$ is the scale parameter, and ε_t is an i.i.d. innovation with distribution function F_ε and zero location.

There are several ways to define the scale parameter. A commonly used one is the standard deviation, which is based on moments. Despite its widespread usage, the standard deviation is sensitive to the presence of outliers and may even be undefined or infinite for distributions with heavy tails. This sensitivity arises from the fact that the standard deviation has a zero breakdown point, meaning that even a single outlier can significantly distort the estimate (see Huber and Ronchetti, 2009, Ch. 5). In contrast, scale estimators based on interquantile ranges have non-zero breakdown points, making them more robust to outliers and extreme values, which are common in financial return data.[1]

With the location-scale representation in (5), the pth quantile of the unconditional return distribution is $Q_{r_t}(p) = \mu + sF_\varepsilon^{-1}(p)$. Let $0 < p < 0.5$ be a chosen quantile level. Following Taylor (2005), consider the quantile at the $(1-p)$th level as a natural and compelling counterpart. Even when μ is unknown, we can obtain s as

$$s = \frac{Q_{r_t}(1-p) - Q_{r_t}(p)}{c_p}, \tag{6}$$

where $c_p = F_\varepsilon^{-1}(1-p) - F_\varepsilon^{-1}(p)$ is a scaling term. Note that when $r_t \sim N(\mu, \sigma^2)$, then (6) yields $s = \sigma$. The quantity $\text{IQR} = Q_{r_t}(0.75) - Q_{r_t}(0.25)$ is the well-known *interquartile range*, which takes the form in (6) by setting $c_p = 1$. Another measure that fits this format is the Pearson and Tukey (1965) scale measure that uses $p = 0.05$ and $c_p = q_0(1-p) - q_0(p)$, where $q_0(p)$ is the pth quantile of the standard normal distribution. See Taylor (2005) and Kotz and van Dorp (2005, Table 1) for other examples.

[1] Kim and White (2004) also advocate using quantiles to estimate characteristics of return distributions, focusing on robust measures for capturing skewness and kurtosis. Through simulations, they demonstrate that empirical moment-based estimators are significantly more sensitive to outliers. See also White et al. (2010) and Ghysels (2014).

2.2 Conditional dynamics

We extend the basic specification in (5) to allow the location and scale parameters to be conditionally dynamic so that

$$r_t = \mu_t + s_t \varepsilon_t, \tag{7}$$

where μ_t and $s_t > 0$ are functions of past information, $\mathcal{I}_{t-1}$. This structure mirrors the typical GARCH framework, where both the conditional mean and the conditional standard deviation evolve over time based on past information. In the following, we focus primarily on first-order specifications when modeling the dynamics in μ_t and s_t. While higher lag orders can be taken into account if required, such specifications are widely favored and commonly used in practice.

In some applications, it may be reasonable to set μ_t to zero or to assume that $\mu_t = \mu$, as done in Taylor (2019). However, autocorrelation in financial returns is often non-negligible (Kuester et al., 2006). This property can be incorporated into our quantile-based modeling framework by allowing the returns to have a time-varying conditional median of the form $\mu_t = Q_{r_t}(0.5 \,|\, \mathcal{I}_{t-1})$, which may be captured by a quantile autoregression (QAR) model, as proposed by Koenker and Xiao (2006). For example, a first-order QAR model is formulated as

$$Q_{r_t}(0.5 \,|\, \mathcal{I}_{t-1}) = \mu + \phi r_{t-1}, \tag{8}$$

where μ represents a baseline median, and ϕ captures the impact of r_{t-1} on the current period's median.

Our analysis concentrates on downside market risk. Accordingly, we define $0 < \tau \leq 0.05$, where τ represents the typical left-tail region of interest in financial risk management, capturing extreme losses. We then set $\tau \leq p < 0.5$, where p specifies the quantile levels used to construct the scale measure. Consider the conditional version of (6) written here as

$$s_t = s_t(p) = \frac{Q_{r_t}(1-p\,|\,\mathcal{I}_{t-1}) - Q_{r_t}(p\,|\,\mathcal{I}_{t-1})}{c_p}, \tag{9}$$

where $c_p = F_\varepsilon^{-1}(1-p) - F_\varepsilon^{-1}(p)$ is a time-invariant scaling term, as before.[2] It is interesting to note here again that s_t is invariant to the specific process governing μ_t in (7). We can then express the model specification as

$$\begin{aligned} r_t &= \mu_t + s_t^* \varepsilon_t^*, \\ s_t^* &= Q_{y_t}(1-p\,|\,\mathcal{I}_{t-1}) - Q_{y_t}(p\,|\,\mathcal{I}_{t-1}), \end{aligned} \tag{10}$$

where $y_t = r_t - \mu_t$ and $\varepsilon_t^* = y_t/s_t^* = \varepsilon_t/c_p$ is simply a rescaled version of ε_t. The τth conditional quantile of $y_t = s_t^*\varepsilon_t^*$ is given by $Q_{y_t}(\tau\,|\,\mathcal{I}_{t-1}) = s_t^* F_{\varepsilon^*}^{-1}(\tau)$, where $F_{\varepsilon^*}^{-1}(\tau)$ is a constant. Similar to the conditional standard deviation in a GARCH model, s_t^* serves as a common factor across quantiles, ensuring coherence and building dependence between them in a parsimonious manner.

To model the dynamics of s_t^*, we draw a parallel with the absolute value GARCH model of Taylor (1986, Section 3.6) and Schwert (1990), which captures the conditional standard deviation using past absolute deviations. We assume

$$s_t^* = \omega + \beta s_{t-1}^* + \gamma |y_{t-1}|, \tag{11}$$

so that the scale of returns adjusts dynamically in a manner similar to the absolute value GARCH framework. Observing that $s_{t-j}^* F_{\varepsilon^*}^{-1}(k) = Q_{y_{t-j}}(k\,|\,\mathcal{I}_{t-j-1})$, we adopt the approach of Xiao and Koenker (2009), who use an absolute value GARCH structure, to express the

[2] As Taylor (2005) observes, estimating scale (or volatility) using the interval between tail quantiles shares similarities with range-based volatility estimation (e.g., Parkinson, 1980; Garman and Klass, 1980; Alizadeh et al., 2002). This class of methods relies on the difference between the highest and lowest log prices, which essentially correspond to the quantiles at $1-p=1$ and $p=0$, respectively.

two key quantiles in (10) with the following global SAV CAViaR structure:

$$Q_{y_t}(k \,|\, \mathcal{I}_{t-1}) = \omega(k) + \big(\beta + \gamma|\varepsilon^*_{t-1}|\big)Q_{y_{t-1}}(k \,|\, \mathcal{I}_{t-2}), \tag{12}$$

for both $k = p$ and $k = 1-p$, subject to $\omega(p) < \omega(1-p)$, $\beta \geq 0$, and $\gamma \geq 0$. These restrictions ensure non-crossing quantiles and guarantee $s^*_t > 0$ in (10), provided the recursion begins with $Q_{y_1}(p \,|\, \mathcal{I}_0) < Q_{y_1}(1-p \,|\, \mathcal{I}_0)$. Here, the persistence parameters β and γ are global, meaning they do not vary with k, while $\omega(k) = \omega F^{-1}_{\varepsilon^*}(k)$ is local and dependent on the quantile level k.

As an alternative to (11), we also consider a threshold-based specification, similar to the threshold GARCH model of Zakoïan (1994):

$$s^*_t = \omega + \beta s^*_{t-1} + \big(\gamma_+ \mathbb{1}\{y_{t-1} > 0\} + \gamma_- \mathbb{1}\{y_{t-1} \leq 0\}\big)|y_{t-1}|,$$

which allows s^*_t to respond differently to positive and negative values of y_{t-1}. This leads to the following global AS CAViaR structure:

$$Q_{y_t}(k \,|\, \mathcal{I}_{t-1}) = \omega(k) + \Big(\beta + \big(\gamma_+ \mathbb{1}\{y_{t-1} > 0\} + \gamma_- \mathbb{1}\{y_{t-1} \leq 0\}\big)|\varepsilon^*_{t-1}|\Big)Q_{y_{t-1}}(k \,|\, \mathcal{I}_{t-2}), \tag{13}$$

for both $k = p$ and $k = 1 - p$, subject to $\omega(p) < \omega(1 - p)$, $\beta \geq 0$, $\gamma_+ \geq 0$, and $\gamma_- \geq 0$. Here, β, γ_+, and γ_- are global parameters. This specification allows an asymmetric response in the conditional quantile process, capturing the differing impacts of positive and negative past shocks. Empirical studies by Nelson (1991), Glosten et al. (1993), and Engle and Ng (1993), among others, provide strong evidence that asymmetry is crucial for accurately modeling volatility dynamics, as negative stock returns are typically associated with greater increases in return volatility. This justifies the inclusion of an asymmetric term in financial time series models such as the AS CAViaR. For simplicity, the notation in (3), (4), (8), (12), and (13) omits the dependence on the involved parameters.

For estimation purposes, we recommend the following sequential strategy: (i) estimate the parameters of (8) and obtain the fitted values $\hat{\mu}_t = \hat{Q}_{r_t}(0.5 \,|\, \mathcal{I}_{t-1})$, and then (ii) estimate the parameters of the CAViaR specifications with μ_t replaced by $\hat{\mu}_t$. While this approach may not be as statistically efficient as joint estimation, it offers the advantage of being simpler and more numerically robust. We begin with the estimation of the QAR model. Let $\boldsymbol{\theta}_k$ represent the parameter vector for (8) with $k = 0.5$. The estimate $\hat{\boldsymbol{\theta}}_k$ can be obtained by solving the minimization problem

$$\min_{\boldsymbol{\theta}_k} \sum_t \rho_k\big[r_t - Q_{r_t}(k \,|\, \mathcal{I}_{t-1})\big], \tag{14}$$

where $\rho_k[u] = u(k - \mathbb{1}\{u \leq 0\})$ is the quantile regression "check" loss function (Koenker and Bassett, 1978).

The estimation of (12) and (13) proceeds with restricted quantile regression. When fitting these CAViaR specifications to data, a choice needs to be made for the initial value $Q_{y_1}(k \,|\, \mathcal{I}_0)$. A simple and reasonable approach is to set $Q_{y_1}(k \,|\, \mathcal{I}_0) = \hat{Q}_y(k)$, the corresponding sample quantile. This choice ensures that all subsequent conditional quantiles will satisfy the non-crossing condition, both in and out of sample. For instance, in the case of (12), the parameter estimates for $\boldsymbol{\theta} = \big(\omega(p), \omega(1-p), \beta, \gamma\big)'$ are found by solving:

$$\begin{aligned} &\min_{\boldsymbol{\theta}} \sum_{k \in \{p, 1-p\}} \sum_t \rho_k \left[y_t - Q_{y_t}(k \,|\, \mathcal{I}_{t-1})\right] \\ &\text{subject to} \begin{cases} \omega(p) < \omega(1-p), \\ \beta \geq 0,\ \gamma \geq 0, \end{cases} \end{aligned}$$

with $Q_{y_1}(k \,|\, \mathcal{I}_0) = \hat{Q}_y(k)$, for $k \in \{p, 1-p\}$. Note that the constraints apply only to the model parameters, without involving the data. The estimation of (13) follows a similar procedure, with the constraint on γ replaced by $\gamma_+ \geq 0$ and $\gamma_- \geq 0$.

With the estimated parameters, the fitted value of $s_t^* = \hat{s}_t^*(p)$ in (10) is calculated as

$$\hat{s}_t^*(p) = \hat{Q}_{y_t}(1 - p \,|\, \mathcal{I}_{t-1}) - \hat{Q}_{y_t}(p \,|\, \mathcal{I}_{t-1}).$$

Then, using the fitted location $\hat{\mu}_t$ and scale numerator $\hat{s}_t^*(p)$, it is natural to use the unconditional quantiles of $\hat{\varepsilon}_t^*(p) = (r_t - \hat{\mu}_t)/\hat{s}_t^*(p)$ as estimates for $F^{-1}_{\varepsilon^*(p)}(\tau)$, since we are not assuming any particular distribution for ε_t in (7). The estimated left-tail τ-quantile of the conditional return distribution is then given by

$$\hat{Q}_{r_t}(\tau, p \,|\, \mathcal{I}_{t-1}) = \hat{\mu}_t + \hat{s}_t^*(p)\hat{F}^{-1}_{\varepsilon^*(p)}(\tau), \tag{15}$$

for $0 < \tau \leq 0.05$.

2.3 Estimating VaR

We estimate the VaR in (1) based on the left-tail quantile of the conditional return distribution. In our quantile-based framework, the VaR estimate for a given probability level α is defined as

$$\widehat{\text{VaR}}_{r_t}(\alpha, p) = \hat{Q}_{r_t}(\alpha, p \,|\, \mathcal{I}_{t-1}), \tag{16}$$

where $\hat{Q}_{r_t}(\alpha, p \,|\, \mathcal{I}_{t-1})$ is derived from the dynamic quantile estimate in (15), evaluated at $\tau = \alpha$.

While the VaR estimate in (16) can be computed for a given value of p, relying on a single scale-defining quantile level may lead to inefficiencies. Since the seminal work of Pearson and Tukey (1965), it has been well understood that using the interval between symmetric tail quantiles to estimate standard deviation is influenced by the skewness and kurtosis of the distribution. Furthermore, as early as Mosteller (1946), it was suggested that interquantile range-based estimates of standard deviation could be improved by considering multiple quantile pairs. Building on these insights, we extend this principle to VaR estimation by computing estimates across multiple quantile levels and aggregating the results using an averaging scheme.

Specifically, we define a set of scale-defining quantile levels $\mathcal{P}$ and compute VaR for each $p \in \mathcal{P}$. The final VaR estimate is then given by

$$\widehat{\mathrm{VaR}}_{r_t}(\alpha) = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \widehat{\mathrm{VaR}}_{r_t}(\alpha, p), \tag{17}$$

where $|\mathcal{P}|$ denotes the number of quantile levels used. For our implementation, we use the set $\mathcal{P} = \{0.05, 0.10, 0.15, 0.20, 0.25\}$. While this choice is somewhat ad hoc, it corresponds to the relatively more robust values reported in Huber and Ronchetti (2009, Exhibit 5.4) for estimating scale using interquantile ranges.

We also considered alternative methods, such as using a single quantile level or aggregating via the median across $\mathcal{P}$. However, simulations indicate that averaging yields greater stability and lower overall estimation error. This approach mitigates fluctuations in tail behavior and enhances out-of-sample performance, particularly for heavy-tailed distributions. A detailed comparison of these methods is provided in Section A of the Supplementary material.

2.4 Approximating ES

The ES in (2) represents the conditional expectation of loss, given that the loss has exceeded the VaR threshold at a specified probability level α. As shown by Acerbi and Tasche (2002), among others, ES can be expressed as

$$\mathrm{ES}_{r_t}(\alpha) = \frac{1}{\alpha} \int_0^{\alpha} Q_{r_t}(\tau \,|\, \mathcal{I}_{t-1})\, d\tau, \tag{18}$$

where $Q_{r_t}(\tau \,|\, \mathcal{I}_{t-1})$ represents the conditional quantile for $\tau \leq \alpha$.

To compute this integral in practice, we approximate it using a sum over N equally spaced subdivisions between 0 and α, similar to a Riemann sum approximation:

$$\widetilde{\mathrm{ES}}_{r_t}(\alpha, p, N) = \frac{1}{N} \sum_{i=1}^{N} \hat{Q}_{r_t}(\tau_i, p \,|\, \mathcal{I}_{t-1}),$$

where $\tau_i = i\alpha/N$ for $i = 1, \ldots, N$, and each $\hat{Q}_{r_t}(\tau_i, p \,|\, \mathcal{I}_{t-1})$ is the corresponding quantile estimate from (15).

As with VaR, we compute ES for several values of p to obtain a more robust estimate. Specifically, for each fixed N, we take the average ES across different quantile levels p:

$$\widehat{\mathrm{ES}}_{r_t}(\alpha, N) = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \widetilde{\mathrm{ES}}_{r_t}(\alpha, p, N).$$

This averaging process reduces estimation errors associated with any single p, leading to a more efficient and reliable ES estimate. The order of operations is important: by first approximating the integral over τ before averaging across p, the quantiles used in each $\widetilde{\mathrm{ES}}_{r_t}(\alpha, p, N)$ share a consistent scale structure. Averaging across p only afterward preserves the internal coherence of each quantile specification and avoids mixing quantiles constructed under differing scale definitions.

Rather than fixing N a priori, we employ an adaptive selection method. We initialize with $N = 4$ and iteratively increase it, computing ES at each step. The iteration stops when successive averaged ES estimates stabilize within a predefined tolerance level ϵ:

$$\left| \widehat{\mathrm{ES}}_{r_t}(\alpha, N) - \widehat{\mathrm{ES}}_{r_t}(\alpha, N-1) \right| < \epsilon,$$

where we set $\epsilon = 0.0001$ in our implementation. This ensures sufficient numerical accuracy while avoiding unnecessary computations. The stabilized ES estimate is denoted $\widehat{\mathrm{ES}}_{r_t}(\alpha)$.

An important advantage of the QbSD approach is its ability to prevent inconsistencies common in tail risk estimation. Specifically, it avoids: (i) quantile crossings, (ii) crossings among ES values at different probability levels, and (iii) crossings between quantiles and their corresponding ES values. This method ensures valid, well-ordered risk estimates across all probability levels α, even in the extreme tails of the distribution.

3 Simulation experiments

In this section, we investigate the accuracy of the proposed QbSD approach with respect to the one-step-ahead VaR and ES forecasts at probability level α. The performance of the QbSD models is evaluated using mean absolute error (MAE) and root mean squared error (RMSE), and compared to several benchmark models, including GARCH and joint VaR-ES models. The analysis also assesses the models' ability to capture skewness, heavy tails, and leverage effects in financial returns.

3.1 Simulation design

The data-generating process (DGP) is the asymmetric power ARCH (APARCH) model of Ding et al. (1993), specified as

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t, \\ \sigma_t^\delta &= \omega + \beta \sigma_{t-1}^\delta + \gamma(|r_{t-1}| - \theta r_{t-1})^\delta, \end{aligned} \tag{19}$$

where $\omega, \beta, \gamma, \delta > 0$ and $-1 < \theta < 1$. The innovations ε_t follow the skewed t-distribution of Hansen (1994). This model nests several well-known specifications as special cases. In particular, when $\delta = 2$ and $\theta = 0$, the model reduces to a standard GARCH model. When $\delta = 1$, it becomes the absolute value GARCH model of Taylor (1986, Section 3.6) and Schwert (1990), which implies direct CAViaR representations for the conditional quantiles (Xiao and

Koenker, 2009). Furthermore, when $\theta \neq 0$, the model implies an asymmetric response of volatility to past returns. When $\theta > 0$, this asymmetry takes the form of a leverage effect, whereby negative returns have a larger impact on future volatility than positive returns of the same magnitude (Black, 1976; Christie, 1982; French et al., 1987).

The density function of the innovation term ε_t appearing in (19) is defined as

$$f_\varepsilon(x; v, \lambda) = \begin{cases} bc\left(1 + \frac{1}{v-2}\left(\frac{bx+a}{1-\lambda}\right)^2\right)^{-(v+1)/2}, & \text{if } x \leq -a/b \\ bc\left(1 + \frac{1}{v-2}\left(\frac{bx+a}{1+\lambda}\right)^2\right)^{-(v+1)/2}, & \text{if } x > -a/b \end{cases} \tag{20}$$

where the constants a, b, and c are themselves defined as

$$a = 4\lambda c\frac{v-2}{v-1}, \quad b = \sqrt{1 + 3\lambda^2 - a^2}, \quad c = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{\pi(v-2)}\Gamma(\frac{v}{2})},$$

and the parameters $v > 2$ and $-1 < \lambda < 1$ represent the degrees of freedom and the asymmetry of the distribution, respectively. The skewed t-distribution has a mean of zero and unit variance. When $\lambda = 0$, it reduces to a standardized version of the traditional Student-t distribution. A positive λ implies right skewness. Financial returns typically have a higher probability of large negative returns than large positive ones, corresponding to left skewness ($\lambda < 0$).

The distribution in (20) is piecewise-defined around its mode at $x = -a/b$, which separates the left and right parts of the density and plays an important role in the derivation of VaR and ES. Jondeau and Rockinger (2003) show that the associated cumulative distribution function (CDF) is defined by

$$F_\varepsilon(x; v, \lambda) = \begin{cases} (1-\lambda)F_{\mathcal{T}}\left(\frac{bx+a}{1-\lambda}\sqrt{\frac{v}{v-2}}; v\right), & \text{if } x \leq -a/b \\ (1+\lambda)F_{\mathcal{T}}\left(\frac{bx+a}{1+\lambda}\sqrt{\frac{v}{v-2}}; v\right) - \lambda, & \text{if } x > -a/b \end{cases}$$

where $F_{\mathcal{T}}(x;v)$ is the CDF of the Student-t distribution with v degrees of freedom, and the quantile function of the skewed t-distribution is given by

$$F_{\varepsilon}^{-1}(u;v,\lambda)=\begin{cases}\frac{1}{b}\left((1-\lambda)\sqrt{\frac{v-2}{v}}F_{\mathcal{T}}^{-1}\left(\frac{u}{1-\lambda};v\right)-a\right), & \text{if } u\leq\frac{1-\lambda}{2}\\ \frac{1}{b}\left((1+\lambda)\sqrt{\frac{v-2}{v}}F_{\mathcal{T}}^{-1}\left(\frac{u+\lambda}{1+\lambda};v\right)-a\right), & \text{if } u>\frac{1-\lambda}{2}.\end{cases}$$

Under this DGP, both the VaR and ES of r_t are proportional to σ_t. The true one-step-ahead VaR forecast is $\mathrm{VaR}_{r_{t+1}}(\alpha;v,\lambda)=\sigma_{t+1}F_{\varepsilon}^{-1}(\alpha;v,\lambda)$ and the true one-step-ahead ES forecast is $\mathrm{ES}_{r_{t+1}}(\alpha;v,\lambda)=\sigma_{t+1}\mathrm{ES}_{\varepsilon}(\alpha;v,\lambda)$, where the ES for Hansen's (1994) skewed t-distribution is given by Patton et al. (2019) as[3]

$$\mathrm{ES}_{\varepsilon}(\alpha;v,\lambda)=\begin{cases}\mathrm{ES}_{\varepsilon}^{*}(\alpha;v,\lambda), & \text{if } F_{\varepsilon}^{-1}(\alpha;v,\lambda)\leq -a/b\\ \frac{1-\alpha}{\alpha}\mathrm{ES}_{\varepsilon}^{*}(1-\alpha;v,-\lambda), & \text{if } F_{\varepsilon}^{-1}(\alpha;v,\lambda)> -a/b\end{cases}\tag{21}$$

with

$$\mathrm{ES}_{\varepsilon}^{*}(\alpha;v,\lambda)=\frac{\tilde{\alpha}}{\alpha}(1-\lambda)\Big(-\frac{a}{b}+\frac{1-\lambda}{b}\mathrm{ES}_{\mathcal{T}}(\tilde{\alpha};v)\Big)$$

and

$$\tilde{\alpha}=F_{\varepsilon}\left(\frac{b}{1-\lambda}\Big(F_{\varepsilon}^{-1}(\alpha;v,\lambda)+\frac{a}{b}\Big);v,0\right).$$

Here $\mathrm{ES}_{\mathcal{T}}(\alpha;v)$ is the ES for a Student-t distribution, standardized to have unit variance, given by

$$\mathrm{ES}_{\mathcal{T}}(\alpha;v)=-\frac{1}{\alpha}f_{\mathcal{T}}(q_\alpha;v)\Big(\frac{v+q_\alpha^2}{v-1}\Big)\sqrt{\frac{v-2}{v}},$$

where $f_{\mathcal{T}}(x;v)$ and $q_\alpha=F_{\mathcal{T}}^{-1}(\alpha;v)$ are, respectively, the density and α-quantile of the

[3]The expression in Appendix B.4 of Patton et al. (2019) contains some typos. It can be verified, either analytically or by simulation, that (21) is the correct ES expression for Hansen's skewed t-distribution. In the second branch of (21), the parameters are flipped to $(v,-\lambda)$ (so the constants change to a_2, b_2 accordingly) when evaluating $\mathrm{ES}_{\varepsilon}^{*}$. The prefactor $(1-\alpha)/\alpha$ converts that right-tail ES at level $1-\alpha$ back to the desired left-tail level α.

Student-t distribution with v degrees of freedom (Broda and Paolella, 2011; Dobrev et al., 2017).

In the DGP outlined in (19), we set the parameters as follows: $\omega = 0.05$, $\beta = 0.85$, $\gamma = 0.10$, and $\delta = 1.5$, while allowing θ to take values in $\{0, 0.5\}$. For the skewed t-distribution in (20), we select $v \in \{5, 20\}$ to represent different tail thicknesses, with $v = 5$ corresponding to heavy tails and $v = 20$ representing thinner tails. We use $\lambda \in \{0, -0.5\}$ to capture different levels of skewness, with $\lambda = 0$ indicating no skewness and $\lambda = -0.5$ implying left skewness. The sample size is set to $T = 1250$, which corresponds to roughly five years of daily returns and matches the primary rolling-window length used in our empirical application. The results presented in the main text focus on this baseline case; additional simulations for $T \in \{250, 2500\}$ are reported in Section B of the Supplementary material. We consider three probability levels for the forecasts: $\alpha \in \{0.01, 0.025, 0.05\}$. The choice of $\delta = 1.5$ allows us to explore a setting where the power parameter lies between the standard GARCH model ($\delta = 2$) and the absolute value GARCH model ($\delta = 1$).[4] We set $\theta \in \{0, 0.5\}$ to explore the role of asymmetry in volatility dynamics. As noted earlier, $\theta = 0$ yields a symmetric response, while $\theta = 0.5$ introduces a leverage effect.

For each DGP configuration, we evaluate the forecasting accuracy by computing the MAE and the RMSE across all 1,000 generated return series. These metrics allow us to compare the performance of the models in capturing the true one-step-ahead VaR and ES forecasts. In our simulations, we consider the QbSD model with the two global CAViaR specifications in (12) and (13). Consistent with the simulation DGP in (19), all QbSD and benchmark models, presented next, assume a zero location for returns. This assumption simplifies the modeling process, allowing us to focus on volatility and tail risk dynamics.

[4]This value of δ is close to the estimate of 1.43 reported by Ding et al. (1993) in their study of S&P 500 returns.

3.2 Benchmark models

To assess the performance of our QbSD models, we compare them with several established models commonly used in financial risk forecasting. The benchmarks include the VaR and ES forecasts from the standard GARCH model, the asymmetric GJR-GARCH model (Glosten et al., 1993), and the EGARCH model (Nelson, 1991), each with normal innovations. Additionally, for the GARCH and GJR-GARCH models, we consider alternative distributional assumptions for the innovations, including the Student-t distribution and Hansen's (1994) more general skewed t-distribution.[5]

We also include the joint VaR-ES model proposed by Taylor (2019), which employs a pseudo-maximum likelihood estimation approach based on the AL density function for the joint estimation of VaR and ES. The conditional AL density function is given by

$$f(r_t) = \frac{(\alpha - 1)}{\text{ES}_t} \exp\left(\frac{(r_t - \text{VaR}_t)(\alpha - \mathbb{1}\{r_t \leq \text{VaR}_t\})}{\alpha \text{ES}_t}\right),$$

where VaR_t is modeled using either the SAV or AS CAViaR specifications from expressions (3) and (4), with y_t replaced by r_t under the mean zero assumption. The complete model includes two alternative specifications for the dynamics of the ES:

$$\text{ES}_t = (1 + \exp(\gamma_0)) \text{VaR}_t \tag{22}$$

or

$$\text{ES}_t = \text{VaR}_t - x_t, \tag{23}$$

[5]We consider the EGARCH model only with normal innovations, as it tends to exhibit instability when applied with t-distributed errors (Nelson, 1991, p. 365).

where x_t follows a dynamic updating process:

$$x_t = \begin{cases} \gamma_0 + \gamma_1(\text{VaR}_{t-1} - r_{t-1}) + \gamma_2 x_{t-1}, & \text{if } r_{t-1} \leq \text{VaR}_{t-1} \\ x_{t-1}, & \text{otherwise.} \end{cases}$$

This formulation provides a smooth adjustment to the magnitude of exceedances beyond the VaR, dynamically updating based on past returns and exceedances.

To differentiate between the two AL-based specifications, we use the notation "$\text{AL}_{\text{Mult.}}$" for the multiplicative version in (22), where ES is modeled as a multiple of VaR, and "AL_{AR}" for the autoregressive version in (23). For these AL-based models, we further indicate the choice of VaR specification using the suffixes "-SAV" for the SAV CAViaR model in (3) and "-AS" for the AS CAViaR model in (4). For example, $\text{AL}_{\text{Mult.}}$-SAV refers to the multiplicative AL-based model with the SAV CAViaR specification. The QbSD models, in contrast, rely on global CAViaR specifications, where the scale dynamics are shared across quantiles. Here, the suffix "-gSAV" denotes the global SAV CAViaR specification (12), while "-gAS" refers to the global AS CAViaR specification (13). For example, QbSD-gSAV refers to the QbSD model with the global SAV CAViaR specification.

In addition to the quantile-based models, we also consider the one-factor GAS model proposed by Patton et al. (2019), which provides an alternative framework for jointly modeling VaR and ES. This model is based on the following member of the Fissler and Ziegel (2016) class of loss functions:

$$L_{\text{FZ0}}(r_t, \text{VaR}_t, \text{ES}_t) = -\frac{1}{\alpha \text{ES}_t} \mathbb{1}\{r_t \leq \text{VaR}_t\}(\text{VaR}_t - r_t) + \frac{\text{VaR}_t}{\text{ES}_t} + \log(-\text{ES}_t) - 1.$$

The FZ_0 loss function is strictly consistent for the pair (VaR, ES). Although the loss itself is not homogeneous of degree zero—due to the $\log(-\text{ES}_t)$ term—the differences in loss values between competing forecasts are. This ensures that forecast rankings based on average loss are invariant to positive scaling of both returns and risk measures.

The joint specification for VaR and ES is

$$\mathrm{VaR}_t = \zeta \exp(\kappa_t), \quad \mathrm{ES}_t = \xi \exp(\kappa_t), \quad \text{with } \xi < \zeta < 0,$$

where the common factor κ_t evolves dynamically as

$$\kappa_t = \omega + \beta \kappa_{t-1} + \gamma g_{t-1},$$

and the score function g_t, derived from the FZ0 loss function, is given by

$$g_t = -\frac{1}{\mathrm{ES}_t}\left(\frac{1}{\alpha}\mathbb{1}\{r_t \leq \mathrm{VaR}_t\} r_t - \mathrm{ES}_t\right).$$

This approach dynamically updates both VaR and ES based on the information in past losses.[6] Following Patton et al. (2019), we set $\omega = 0$ for identification and estimate the model by minimizing the average FZ0 loss.

These competing models offer a range of benchmarks for evaluating the forecasting performance of the QbSD models. In the next section, we present the simulation results and assess how well each model captures key features of financial returns across the various DGP configurations. It is interesting to note that none of the models exactly matches the DGP in (19), which makes the forecasting task more challenging and offers a stricter test of adaptability.

[6]Since the Hessian of the FZ0 loss is constant, the scaling matrix in the GAS framework of Creal et al. (2013) can be set to one without loss of generality. This simplifies estimation while preserving the model's essential dynamics.

3.3 Numerical results

This section presents the forecasting results for the QbSD and benchmark models across the simulation scenarios. Tables 1–4 report out-of-sample one-step-ahead VaR and ES accuracy (MAE and RMSE) under symmetric and asymmetric volatility dynamics.

Table 1 reports VaR forecast accuracy with no leverage (symmetric volatility dynamics). Parametric GARCH-family models with correctly specified innovations (Student-t under symmetry; skew-t under left skew) lead across configurations. With symmetric innovations ($v = 20, \lambda = 0$), GARCH/GJR with Student-t is best across α (e.g., MAE $0.071, 0.054, 0.044$; RMSE $0.095, 0.073, 0.059$), while QbSD-gSAV is competitive but not top. With left skewness ($v = 20, \lambda = -0.5$), skew-t dominates (e.g., GJR-skew-t MAE $0.098, 0.068, 0.049$; RMSE $0.132, 0.093, 0.069$), with only minor differences between GARCH and GJR. Under heavier tails ($v = 5$), the pattern persists: when $\lambda = 0$, GARCH/GJR-t remains best (MAE $0.103, 0.068, 0.050$; RMSE $0.143, 0.097, 0.072$); when $\lambda = -0.5$, GJR-GARCH with skew-t leads (MAE $0.145, 0.084, 0.054$; RMSE $0.213, 0.134, 0.093$). Across all blocks, QbSD-gSAV improves on AL-based and GAS alternatives but trails the best-specified GARCH variants.

Turning to ES under symmetric volatility (Table 2), the same ordering emerges: matching the innovation distribution to the DGP is pivotal. In the symmetric case ($v = 20, \lambda = 0$), GARCH/GJR-t attains the lowest errors (MAE $0.100, 0.075, 0.060$; RMSE $0.129, 0.099, 0.081$). Introducing left skewness ($v = 20, \lambda = -0.5$) shifts the frontier to skew-t (GJR-skew-t MAE $0.146, 0.105, 0.080$; RMSE $0.191, 0.140, 0.108$). Under heavier tails ($v = 5$), Student-t remains best when $\lambda = 0$ (MAE $0.180, 0.118, 0.086$; RMSE $0.241, 0.162, 0.120$), while skew-t variants lead for $\lambda = -0.5$ (e.g., GJR-skew-t MAE $0.117, 0.173, 0.117$; RMSE $0.174, 0.247, 0.383$). AL-based and GAS models are consistently less accurate.

Table 3 reports VaR accuracy with leverage ($\theta = 0.5$). QbSD-gAS delivers the lowest errors in most configurations featuring skewness and/or heavy tails. For example, at $v =$

$20, \lambda = -0.5$, it is best across α (MAE $0.172, 0.125, 0.096$; RMSE $0.238, 0.183, 0.138$). In the heavy-tailed but symmetric case ($v = 5, \lambda = 0$), QbSD-gAS is generally strongest, with EGARCH edging it at $\alpha = 2.5\%$ (MAE 0.105 vs. 0.110). Under heavy tails with left skewness ($v = 5, \lambda = -0.5$), GJR-skew-t attains the lowest MAE ($0.233, 0.152, 0.108$), with QbSD-gAS close behind. A notable exception is the thin-tailed, symmetric DGP ($v = 20, \lambda = 0$), where EGARCH dominates (MAE $0.105, 0.075, 0.062$; RMSE $0.148, 0.107, 0.086$).

Table 4 presents ES accuracy with leverage. QbSD-gAS leads in most configurations. It is best across α at $v = 20, \lambda = -0.5$ (MAE $0.235, 0.174, 0.140$; RMSE $0.314, 0.238, 0.195$) and at $v = 5, \lambda = 0$ (MAE $0.292, 0.186, 0.136$). In the heavy-tailed, left-skewed case $v = 5, \lambda = -0.5$, GJR-skew-t has the edge (MAE $0.388, 0.263, 0.191$; RMSE $0.526, 0.352, 0.256$). For the thin-tailed symmetric DGP $v = 20, \lambda = 0$, EGARCH is competitive and surpasses QbSD-gAS at higher α (e.g., MAE $0.113, 0.086$ and RMSE $0.157, 0.122$ at 2.5% and 5%, vs. QbSD-gAS $0.125, 0.103$ and $0.165, 0.137$); QbSD-gAS is slightly better at 1% (MAE 0.165 vs. 0.167).

Overall, the evidence suggests a clear division: when volatility responds symmetrically and the innovation distribution is correctly specified (e.g., appropriately heavy-tailed or skewed), parametric GARCH variants set the benchmark for VaR and ES; once leverage/asymmetry matters, allowing asymmetric-slope (gAS) scale dynamics is pivotal—the proposed QbSD-gAS specification is typically strongest, particularly for ES, with EGARCH competitive and occasionally best in thin-tailed symmetric settings. Across these scenarios, AL-based and GAS models remain less accurate. We next examine whether these patterns carry over to international stock index returns.

4 Empirical Application

We evaluate one-day-ahead VaR and ES forecasts for the daily log returns of the S&P 500, DJIA, NASDAQ, EURO STOXX 50, FTSE 100, DAX, CAC 40, and TSX stock indices. The daily adjusted closing prices for these indices were obtained from Yahoo Finance, spanning October 4, 2002, to February 2, 2024, across indices. Due to differences in trading days and data availability, the number of observations initially varies across the series.

To ensure comparability across stock indices, we standardized the sample size to match the index with the shortest available data. Specifically, the EURO STOXX 50 has the shortest sample, starting on October 21, 2002, and containing 5,357 daily observations, ending on February 2, 2024. For consistency, we truncated the data for indices with longer samples by counting backward from February 2, 2024, to retain exactly 5,357 observations for each index. This adjustment implies the following starting dates: October 22, 2002, for the S&P 500, DJIA, and NASDAQ; October 21, 2002, for the EURO STOXX 50; November 18, 2002, for the FTSE 100; January 3, 2003, for the DAX; February 27, 2003, for the CAC 40; and October 4, 2002, for the TSX. As returns are calculated as first differences of log prices, this sample size yields 5,356 daily return observations for all indices.

Figure 1 illustrates the time series of daily log returns (in percentages) for the S&P 500, DJIA, NASDAQ, and EURO STOXX 50 indices, each exhibiting notable spikes in volatility during significant economic events such as the 2008 financial crisis and the COVID-19 pandemic. Figure 2 shows the daily log returns for the FTSE 100, DAX, CAC 40, and TSX indices, which similarly exhibit heightened volatility during periods of financial stress.

We use a rolling window of $R = 1250$ estimation observations (approximately five trading years) to produce one-day-ahead forecasts for VaR and ES. This choice balances sampling variability against responsiveness to structural change. Model parameters are estimated using the most recent 1,250 observations, and forecasts are updated daily as the window moves through the sample period. Given the common sample of 5,356 daily returns, this setup leaves an out-of-sample evaluation period of 4,106 observations. We forecast VaR and ES at probability levels of 1%, 2.5%, and 5%. The 1% and 5% levels are widely used in the literature, whereas the 2.5% level corresponds to the regulatory standard for ES adopted under Basel III. We also examined alternative rolling-window sizes of $R = 250$ and $R = 2500$. To maintain focus in the main text, we present results for the 1,250-observation window, while these alternative specifications appear in Section C of the Supplementary material.

We evaluate a total of twenty-eight competing risk models for forecasting VaR and ES, including: (i) fourteen GARCH-type models (GARCH and GJR-GARCH with normal, Student-t, and Hansen skew-t innovations under zero or AR(1) conditional means, plus EGARCH with normal innovations under zero or AR(1) means), (ii) two GAS variants with zero and AR(1) means (Patton et al., 2019), (iii) eight AL-based models following Taylor (2019) (SAV and AS CAViaR for VaR dynamics, multiplicative and autoregressive specifications for ES dynamics, each under zero or AR(1) means), and (iv) four QbSD models (gSAV and gAS scale dynamics under zero or QAR(1) conditional medians).

In the following tables, we use the prefixes "AR-" and "QAR-" to denote an AR(1) conditional mean and a QAR(1) conditional median, respectively; models without a prefix assume a zero conditional location.

4.1 Evaluation of VaR forecasts

We evaluate the accuracy of VaR forecasts using the model confidence set (MCS) procedure of Hansen et al. (2011), which identifies a subset of models that are not significantly outperformed at a given confidence level. The MCS framework is particularly useful in a multi-model comparison setting, as it does not require selecting a single best model but instead acknowledges model uncertainty by retaining all models that cannot be statistically eliminated.

Following Hansen et al.'s (2011) suggestion, we apply the range-based procedure using the maximum absolute t-statistic, which iteratively removes the model with the largest loss difference relative to the others until only models that cannot be statistically distinguished from the best-performing ones remain. The final MCS consists of models that survive this elimination process at the chosen confidence level.[7]

The test statistic is computed using the quantile score loss, given by

$$S(r_{t+1}, \mathrm{VaR}_{t+1}) = (r_{t+1} - \mathrm{VaR}_{t+1})\left(\alpha - \mathbb{1}\left\{r_{t+1} \leq \mathrm{VaR}_{t+1}\right\}\right), \tag{24}$$

where r_{t+1} is the realized return and VaR_{t+1} is the predicted VaR at a given probability level α. This loss function is asymmetric and penalizes underpredictions more heavily than overpredictions, making it well suited to the evaluation of downside risk.

Tables 5–7 present the MCS results for 1%, 2.5%, and 5% VaR forecasts across the eight international stock indices. Models excluded from the 90% MCS ($\widehat{\mathcal{M}}^*_{90\%}$) for a given index are left blank. The remaining models are ranked by their average quantile score t-statistics; the "#" column counts the number of indices for which a model remains in the MCS, while the "Avg. rank" and "Final rank" summarize overall performance.

For the 1% tail (Table 5), QbSD-gAS attains the best overall rank and appears in the MCS for all eight indices; its closest competitors are GARCH variants with skew-t innovations (GJR-GARCH-skew-t, GARCH-skew-t), followed by GJR-GARCH-t, while AL-based specifications generally trail. At 2.5% (Table 6), QbSD-gAS again ranks first overall (in the MCS for all eight indices), with AL-based models featuring AS dynamics ($\mathrm{AL}_{\mathrm{Mult.}}$-AS, AR-$\mathrm{AL}_{\mathrm{Mult.}}$-AS) close behind; among GARCH-type models, skew-t remains the most competitive and EGARCH sits in the upper middle of the table. By 5% (Table 7), QbSD-gAS still leads, EGARCH moves into second place, $\mathrm{AL}_{\mathrm{Mult.}}$-AS and the QAR-driven QbSD-gAS (QAR-QbSD-gAS) remain prominent, and skew-t GARCH variants continue to perform well.

Across probability levels, the symmetric QbSD specification (QbSD-gSAV) underperforms its asymmetric counterpart, underscoring the value of allowing an asymmetric scale

[7]Specifically, we base the MCS on the $T_{R,\mathcal{M}}$ test statistic developed by Hansen et al. (2011, Section 3.1.2) and compute it with 1,000 bootstrap iterations using the R package 'MCS' (Bernardi and Catania, 2018).

response in practice. Within the GARCH family, skew-t innovations generally improve performance at tighter tails (1% and 2.5%), whereas the strong showing of EGARCH at 5% suggests that volatility asymmetry can partly compensate when attention shifts away from the far tail.

While these results provide important insights into tail-risk quantile prediction, a fuller assessment requires joint evaluation of VaR and ES, which we consider next.

4.2 Joint evaluation of VaR and ES forecasts

To jointly evaluate VaR and ES forecasts, we use the AL log score proposed by Taylor (2019). This scoring rule is specifically designed to assess both VaR and ES, reflecting their joint elicitability. The AL log score is given by

$$S(r_{t+1}, \mu_{t+1}, \mathrm{VaR}_{t+1}, \mathrm{ES}_{t+1}) = -\log\left(\frac{\alpha - 1}{\mathrm{ES}_{t+1} - \mu_{t+1}}\right) - \frac{(r_{t+1} - \mathrm{VaR}_{t+1})(\alpha - \mathbb{1}\{r_{t+1} \leq \mathrm{VaR}_{t+1}\})}{\alpha(\mathrm{ES}_{t+1} - \mu_{t+1})}, \tag{25}$$

where μ_{t+1} is the model's conditional mean and $\mathrm{ES}_{t+1} < \mu_{t+1}$ is required. This proper scoring rule penalizes both the incidence and severity of forecast errors in the left tail, encouraging accurate joint predictions of VaR and ES.

Tables 8–10 summarize the MCS rankings under AL log scores. At the 1% tail (Table 8), the best overall performer is AR-GJR-GARCH-skew-t, with QbSD-gAS a close second; both appear in the MCS for all eight indices. Other skew-t GARCH variants (GARCH-skew-t, GJR-GARCH-skew-t) are also highly competitive, whereas AL-based specifications sit further down the table. At 2.5% (Table 9), leadership shifts to QbSD-gAS, which ranks first overall and is included in the MCS for all eight indices. AL-based models with asymmetric VaR dynamics (AS CAViaR), namely $\mathrm{AL}_{\mathrm{Mult.}}$-AS and AR-$\mathrm{AL}_{\mathrm{Mult.}}$-AS, are close followers, with skew-t GARCH entries forming the next tier. By 5% (Table 10), QbSD-gAS again takes the top spot (in the MCS for all indices), followed by AL_{AR}-AS and $\text{AL}_{\text{Mult.}}$-AS; the QAR-driven QAR-QbSD-gAS also ranks near the front. Skew-t GARCH specifications remain competitive, while EGARCH occupies an upper-mid-table position.

Taken together, these results indicate that joint VaR-ES forecasting favors models that accommodate asymmetric tail dynamics. QbSD-gAS is consistently among the best across tail levels and leads at 2.5% and 5%, while skew-t GARCH variants are particularly strong at the most extreme tail (1%). AL-based models with AS CAViaR VaR dynamics provide robust competitors at 2.5% and 5%, reinforcing the importance of allowing asymmetry in the VaR/ES system.

5 Conclusion

This paper introduced QbSD for forecasting VaR and ES: a semiparametric, distribution-free approach that models conditional scale via restricted quantile regression and accommodates skewness, heavy tails, and leverage—features that intensify during periods of market stress.

Across simulations and international equity indices, a consistent picture emerges. In simulation designs without leverage, parametric GARCH models with heavy-tailed or skewed innovation distributions set a high benchmark for both VaR and ES; Student-t and skew-t specifications are especially effective. With leverage, allowing asymmetric-slope quantile dynamics is pivotal: QbSD-gAS is frequently among the best, particularly for ES, while for VaR, EGARCH and skew-t GJR-GARCH can also be highly competitive depending on the tail level and distributional setting.

In the empirical evaluation with a five-year rolling window ($R = 1250$), QbSD-gAS is the clearest overall performer for VaR under quantile-score MCS: it attains the best average rank at 1%, 2.5%, and 5% across the eight indices, with skew-t GARCH variants forming the next tier and EGARCH rising near the front at 5%. When VaR and ES are assessed jointly via AL log scores, performance at the most extreme tail (1%) tilts toward skew-t GARCH (AR-GJR-GARCH-skew-t), with QbSD-gAS a close second; at 2.5% and 5%, QbSD-gAS leads, and AL-based specifications with AS CAViaR VaR dynamics (both multiplicative and autoregressive ES linkages) are robust followers.

Two practical lessons follow. First, asymmetry helps—but not all asymmetries are equal: QbSD-gAS delivers the most consistent improvements; skew-t innovations help when empirical skewness is material; and EGARCH's volatility asymmetry tends to help when return innovations are near-symmetric with relatively light tails, whereas GJR's benefits appear mainly when combined with skew-t under skewed, heavy-tailed conditions. Second, the preferred specification varies with the tail probability and the evaluation target: skew-t GARCH may perform well at the far tail (1%) under joint VaR-ES scoring, whereas QbSD-gAS provides the most consistent gains across tail levels for VaR and remains among the leaders at 2.5% and 5% in the joint evaluation.

Overall, QbSD—especially with asymmetric-slope (gAS) quantile dynamics—offers a robust, flexible addition to the risk-management toolkit. Its distribution-free construction, strong showing under leverage and during turbulent periods, and straightforward extensibility to log-location-scale settings (e.g., strictly positive losses such as credit losses and insurance claims) make it well suited for practical deployment.

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Data and code availability

The data and code needed to reproduce the results presented in this paper are available at https://github.com/richardluger/QbSD. The code is written in R with C++ functions for computational efficiency.

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Table 1. VaR forecast accuracy when there are no leverage effects

	$v=20, \lambda=0$			$v=20, \lambda=-0.5$			$v=5, \lambda=0$			$v=5, \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.105	0.079	0.061	0.134	0.095	0.071	0.148	0.094	0.066	0.201	0.119	0.080
QbSD-gAS	0.118	0.088	0.070	0.143	0.102	0.079	0.157	0.102	0.075	0.207	0.125	0.086
AL approach												
$AL_{Mult.}$-SAV	0.153	0.101	0.073	0.220	0.136	0.097	0.226	0.129	0.083	0.344	0.180	0.114
AL_{AR}-SAV	0.154	0.102	0.075	0.218	0.137	0.102	0.229	0.132	0.085	0.345	0.187	0.122
$AL_{Mult.}$-AS	0.187	0.115	0.085	0.258	0.157	0.112	0.280	0.151	0.097	0.390	0.214	0.133
AL_{AR}-AS	0.187	0.118	0.089	0.260	0.163	0.119	0.280	0.156	0.103	0.394	0.223	0.142
GAS	0.283	0.204	0.160	0.338	0.236	0.165	0.340	0.225	0.151	0.429	0.252	0.163
GARCH												
Normal	0.086	0.056	**0.044**	0.477	0.303	0.180	0.241	0.087	0.082	0.766	0.367	0.153
Student-t	**0.071**	**0.054**	**0.044**	0.392	0.285	0.195	**0.103**	**0.068**	**0.050**	0.602	0.370	0.218
Skew-t	0.080	0.060	0.047	0.099	**0.068**	**0.049**	0.111	0.073	0.053	0.150	0.087	0.055
GJR-GARCH												
Normal	0.086	0.056	**0.044**	0.477	0.303	0.180	0.241	0.087	0.082	0.765	0.367	0.153
Student-t	**0.071**	**0.054**	**0.044**	0.392	0.285	0.195	**0.103**	**0.068**	**0.050**	0.602	0.369	0.218
Skew-t	0.080	0.059	0.047	**0.098**	**0.068**	**0.049**	0.111	0.073	0.053	**0.145**	**0.084**	**0.054**
EGARCH	0.093	0.066	0.054	0.480	0.306	0.182	0.245	0.099	0.093	0.772	0.367	0.152
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.136	0.101	0.079	0.173	0.123	0.094	0.197	0.126	0.091	0.266	0.161	0.112
QbSD-gAS	0.154	0.115	0.093	0.186	0.134	0.105	0.212	0.137	0.102	0.274	0.170	0.119
AL approach												
$AL_{Mult.}$-SAV	0.221	0.134	0.100	0.311	0.187	0.133	0.332	0.184	0.118	0.524	0.237	0.168
AL_{AR}-SAV	0.225	0.136	0.102	0.313	0.186	0.140	0.326	0.185	0.119	0.516	0.267	0.173
$AL_{Mult.}$-AS	0.259	0.153	0.114	0.349	0.212	0.153	0.396	0.209	0.133	0.539	0.300	0.188
AL_{AR}-AS	0.251	0.156	0.119	0.359	0.219	0.160	0.409	0.211	0.140	0.546	0.318	0.197
GAS	0.380	0.270	0.214	0.438	0.313	0.216	0.493	0.320	0.206	0.609	0.366	0.261
GARCH												
Normal	0.106	0.074	0.060	0.493	0.317	0.194	0.270	0.124	0.123	0.802	0.400	0.193
Student-t	**0.095**	**0.073**	**0.059**	0.410	0.298	0.207	**0.143**	**0.097**	**0.072**	0.625	0.386	0.231
Skew-t	0.106	0.080	0.063	0.132	0.093	0.070	0.153	0.102	0.075	0.224	0.139	0.095
GJR-GARCH												
Normal	0.106	0.074	0.060	0.493	0.317	0.194	0.270	0.125	0.123	0.801	0.399	0.193
Student-t	**0.095**	**0.073**	**0.059**	0.410	0.298	0.207	**0.143**	**0.097**	**0.072**	0.625	0.386	0.231
Skew-t	0.106	**0.080**	0.063	**0.132**	**0.093**	**0.069**	0.153	0.103	0.075	**0.213**	**0.134**	**0.093**
EGARCH	0.121	0.087	0.070	0.502	0.325	0.201	0.292	0.144	0.131	0.814	0.407	0.191

Notes: This table reports the MAE in Panel A and the RMSE in Panel B of various VaR forecasting models. The results are based on 1,000 replications for each configuration of the APARCH model in (19) with innovations following the skewed t-distribution in (20). There are no leverage effects ($\theta = 0$). Bolded values indicate the model with the lowest error.

Table 2. ES forecast accuracy when there are no leverage effects

	$v = 20$, $\lambda = 0$			$v = 20$, $\lambda = -0.5$			$v = 5$, $\lambda = 0$			$v = 5$, $\lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.143	0.106	0.085	0.189	0.134	0.107	0.265	0.164	0.118	0.370	0.222	0.156
QbSD-gAS	0.154	0.117	0.096	0.196	0.142	0.114	0.275	0.171	0.126	0.376	0.228	0.161
AL approach												
$\text{AL}_{\text{Mult.}}$-SAV	0.188	0.125	0.092	0.271	0.173	0.126	0.339	0.196	0.126	0.527	0.281	0.182
AL_{AR}-SAV	0.210	0.142	0.108	0.292	0.196	0.142	0.373	0.219	0.147	0.551	0.315	0.211
$\text{AL}_{\text{Mult.}}$-AS	0.228	0.146	0.108	0.317	0.206	0.148	0.392	0.225	0.145	0.571	0.319	0.210
AL_{AR}-AS	0.233	0.155	0.116	0.327	0.222	0.158	0.413	0.240	0.159	0.583	0.345	0.227
GAS	0.341	0.258	0.203	0.416	0.307	0.220	0.474	0.323	0.221	0.628	0.393	0.256
GARCH												
Normal	0.146	0.094	0.067	0.689	0.501	0.367	0.631	0.322	0.163	1.463	0.898	0.563
Student-t	**0.100**	**0.075**	**0.060**	0.489	0.394	0.315	**0.180**	**0.118**	**0.086**	0.945	0.656	0.470
Skew-t	0.110	0.083	0.067	0.147	0.106	**0.080**	0.190	0.126	0.091	0.122	0.179	0.122
GJR-GARCH												
Normal	0.146	0.095	0.067	0.689	0.501	0.367	0.631	0.322	0.163	1.463	0.898	0.563
Student-t	**0.100**	**0.075**	**0.060**	0.489	0.394	0.315	**0.180**	**0.118**	**0.086**	0.945	0.656	0.470
Skew-t	0.110	0.083	0.067	**0.146**	**0.105**	**0.080**	0.190	0.127	0.092	**0.117**	**0.173**	**0.117**
EGARCH	0.150	0.100	0.075	0.693	0.504	0.370	0.637	0.327	0.169	1.470	0.904	0.568
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.185	0.138	0.112	0.240	0.172	0.138	0.340	0.215	0.157	0.463	0.285	0.205
QbSD-gAS	0.201	0.153	0.126	0.254	0.185	0.149	0.352	0.227	0.168	0.473	0.294	0.211
AL approach												
$\text{AL}_{\text{Mult.}}$-SAV	0.265	0.169	0.126	0.382	0.237	0.173	0.482	0.277	0.179	0.770	0.401	0.262
AL_{AR}-SAV	0.291	0.197	0.147	0.408	0.267	0.195	0.521	0.309	0.209	0.763	0.444	0.304
$\text{AL}_{\text{Mult.}}$-AS	0.313	0.193	0.143	0.429	0.276	0.200	0.536	0.306	0.198	0.760	0.439	0.289
AL_{AR}-AS	0.312	0.210	0.155	0.439	0.300	0.219	0.555	0.330	0.221	0.792	0.480	0.319
GAS	0.452	0.345	0.275	0.536	0.405	0.288	0.716	0.459	0.304	0.921	0.558	0.395
GARCH												
Normal	0.168	0.114	0.085	0.707	0.517	0.382	0.658	0.349	0.192	1.507	0.934	0.596
Student-t	**0.129**	**0.099**	**0.081**	0.515	0.414	0.330	**0.241**	**0.162**	**0.120**	0.982	0.682	0.489
Skew-t	0.145	0.111	0.090	0.192	**0.140**	**0.108**	0.255	0.172	0.127	0.183	0.262	0.411
GJR-GARCH												
Normal	0.168	0.115	0.085	0.707	0.517	0.382	0.658	0.349	0.192	1.506	0.933	0.595
Student-t	**0.129**	**0.099**	**0.081**	0.515	0.414	0.330	**0.241**	**0.162**	**0.120**	0.982	0.682	0.489
Skew-t	0.145	**0.110**	**0.089**	**0.191**	**0.140**	**0.108**	0.256	0.173	0.127	**0.174**	**0.247**	**0.383**
EGARCH	0.184	0.130	0.099	0.717	0.526	0.390	0.678	0.369	0.213	1.524	0.948	0.606

Notes: This table reports the MAE in Panel A and the RMSE in Panel B for various ES forecasting models. See Table 1 for a description of the simulation setup. The results are based on simulations without leverage effects ($\theta = 0$). Bolded values indicate the model with the lowest error.

Table 3. VaR forecast accuracy in the presence of leverage effects

	$v = 20, \lambda = 0$			$v = 20, \lambda = -0.5$			$v = 5, \lambda = 0$			$v = 5, \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.214	0.170	0.139	0.271	0.212	0.166	0.240	0.170	0.130	0.324	0.220	0.159
QbSD-gAS	0.124	0.093	0.075	**0.172**	**0.125**	**0.096**	**0.170**	0.110	**0.080**	0.255	0.159	0.109
AL approach												
$AL_{Mult.}$-SAV	0.260	0.187	0.148	0.347	0.242	0.184	0.326	0.200	0.144	0.473	0.273	0.188
AL_{AR}-SAV	0.261	0.189	0.148	0.351	0.244	0.187	0.330	0.202	0.144	0.480	0.277	0.199
$AL_{Mult.}$-AS	0.198	0.127	0.093	0.299	0.184	0.135	0.312	0.164	0.106	0.482	0.254	0.163
AL_{AR}-AS	0.204	0.131	0.098	0.305	0.193	0.142	0.304	0.169	0.109	0.480	0.267	0.176
GAS	0.391	0.264	0.174	0.605	0.386	0.247	0.447	0.254	0.164	0.751	0.397	0.239
GARCH												
Normal	0.202	0.163	0.135	0.539	0.346	0.215	0.299	0.172	0.149	0.879	0.413	0.180
Student-t	0.200	0.162	0.134	0.440	0.326	0.231	0.221	0.162	0.126	0.682	0.420	0.249
Skew-t	0.205	0.165	0.135	0.211	0.160	0.124	0.226	0.166	0.128	0.239	0.155	0.109
GJR-GARCH												
Normal	0.194	0.156	0.128	0.538	0.345	0.214	0.295	0.164	0.142	0.879	0.413	0.180
Student-t	0.200	0.162	0.134	0.440	0.326	0.231	0.221	0.162	0.126	0.682	0.420	0.249
Skew-t	0.205	0.165	0.135	0.211	0.160	0.124	0.227	0.166	0.128	**0.233**	**0.152**	**0.108**
EGARCH	**0.105**	**0.075**	**0.062**	0.558	0.356	0.213	0.262	**0.105**	0.099	0.901	0.430	0.179
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.283	0.225	0.185	0.363	0.283	0.224	0.330	0.235	0.180	0.451	0.326	0.230
QbSD-gAS	0.166	0.126	0.101	**0.238**	**0.183**	**0.138**	**0.250**	0.178	**0.130**	0.437	0.333	0.222
AL approach												
$AL_{Mult.}$-SAV	0.346	0.252	0.198	0.479	0.335	0.254	0.465	0.282	0.204	0.753	0.412	0.285
AL_{AR}-SAV	0.361	0.255	0.197	0.510	0.341	0.256	0.463	0.287	0.205	0.722	0.409	0.292
$AL_{Mult.}$-AS	0.272	0.175	0.129	0.419	0.267	0.199	0.460	0.235	0.161	0.738	0.397	0.272
AL_{AR}-AS	0.303	0.181	0.136	0.433	0.283	0.209	0.436	0.244	0.162	0.725	0.408	0.284
GAS	0.503	0.345	0.227	0.800	0.547	0.356	0.684	0.350	0.218	1.307	0.627	0.397
GARCH												
Normal	0.264	0.215	0.180	0.622	0.411	0.265	0.377	0.239	0.213	1.027	0.500	0.231
Student-t	0.268	0.216	0.177	0.522	0.390	0.281	0.312	0.229	0.177	0.804	0.500	0.303
Skew-t	0.273	0.220	0.179	0.278	0.211	0.164	0.320	0.234	0.179	**0.326**	**0.212**	**0.150**
GJR-GARCH												
Normal	0.255	0.207	0.173	0.620	0.409	0.263	0.365	0.231	0.209	1.027	0.500	0.231
Student-t	0.268	0.216	0.177	0.522	0.390	0.281	0.312	0.229	0.177	0.804	0.500	0.303
Skew-t	0.274	0.220	0.179	0.278	0.211	0.164	0.319	0.234	0.179	0.317	0.208	0.147
EGARCH	**0.148**	**0.107**	**0.086**	0.647	0.428	0.273	0.334	**0.160**	0.136	1.132	0.584	0.287

Notes: This table reports the MAE in Panel A and the RMSE in Panel B for various VaR forecasting models. See Table 1 for a description of the simulation setup. The results are based on simulations with leverage effects ($\theta = 0.5$). Bolded values indicate the model with the lowest error.

Table 4. ES forecast accuracy in the presence of leverage effects

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.253	0.208	0.179	0.335	0.268	0.223	0.354	0.248	0.193	0.510	0.342	0.257
QbSD-gAS	**0.165**	0.125	0.103	**0.235**	**0.174**	**0.140**	**0.292**	**0.186**	**0.136**	0.445	0.279	0.198
AL approach												
$AL_{Mult.}$-SAV	0.304	0.229	0.189	0.414	0.303	0.244	0.450	0.283	0.208	0.680	0.403	0.287
AL_{AR}-SAV	0.329	0.253	0.201	0.449	0.335	0.266	0.488	0.323	0.225	0.733	0.457	0.341
$AL_{Mult.}$-AS	0.245	0.158	0.116	0.369	0.234	0.179	0.442	0.241	0.157	0.697	0.381	0.256
AL_{AR}-AS	0.262	0.180	0.138	0.403	0.275	0.217	0.454	0.267	0.181	0.752	0.442	0.316
GAS	0.475	0.336	0.232	0.730	0.522	0.329	0.598	0.369	0.235	1.062	0.619	0.365
GARCH												
Normal	0.255	0.207	0.175	0.780	0.566	0.416	0.663	0.367	0.228	1.687	1.033	0.644
Student-t	0.249	0.205	0.175	0.539	0.441	0.357	0.325	0.239	0.188	1.063	0.743	0.532
Skew-t	0.253	0.210	0.178	0.279	0.218	0.179	0.335	0.245	0.192	0.405	0.272	0.197
GJR-GARCH												
Normal	0.247	0.199	0.168	0.779	0.565	0.415	0.661	0.364	0.223	1.687	1.033	0.644
Student-t	0.249	0.205	0.175	0.539	0.441	0.357	0.325	0.239	0.188	1.063	0.743	0.532
Skew-t	0.254	0.210	0.178	0.279	0.218	0.178	0.334	0.245	0.193	**0.388**	**0.263**	**0.191**
EGARCH	0.167	**0.113**	**0.086**	0.804	0.586	0.430	0.682	0.349	0.180	1.711	1.055	0.664
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.338	0.279	0.239	0.450	0.359	0.299	0.473	0.339	0.266	0.675	0.464	0.356
QbSD-gAS	**0.215**	0.165	0.137	**0.314**	**0.238**	**0.195**	**0.392**	**0.266**	**0.204**	0.672	0.465	0.357
AL approach												
$AL_{Mult.}$-SAV	0.404	0.312	0.252	0.580	0.422	0.335	0.641	0.403	0.294	1.071	0.590	0.423
AL_{AR}-SAV	0.443	0.342	0.270	0.664	0.474	0.391	0.682	0.447	0.334	1.110	0.684	0.587
$AL_{Mult.}$-AS	0.328	0.216	0.162	0.513	0.336	0.262	0.619	0.333	0.232	1.015	0.563	0.404
AL_{AR}-AS	0.376	0.250	0.192	0.571	0.404	0.335	0.611	0.380	0.260	1.183	0.675	0.602
GAS	0.623	0.438	0.304	0.991	0.724	0.467	0.936	0.529	0.317	1.809	0.950	0.578
GARCH												
Normal	0.330	0.270	0.230	0.880	0.650	0.487	0.772	0.449	0.295	1.955	1.202	0.759
Student-t	0.331	0.274	0.233	0.643	0.525	0.427	0.456	0.337	0.266	1.251	0.874	0.630
Skew-t	0.340	0.280	0.237	0.367	0.288	0.235	0.468	0.345	0.272	0.562	0.372	0.269
GJR-GARCH												
Normal	0.319	0.260	0.222	0.878	0.648	0.486	0.759	0.437	0.285	1.954	1.202	0.758
Student-t	0.331	0.274	0.233	0.643	0.525	0.427	0.456	0.337	0.266	1.251	0.874	0.630
Skew-t	0.338	0.279	0.237	0.367	0.288	0.261	0.469	0.346	0.271	**0.526**	**0.352**	**0.256**
EGARCH	0.219	**0.157**	**0.122**	0.912	0.676	0.507	0.765	0.421	0.244	2.077	1.309	0.851

Notes: This table reports the MAE in Panel A and the RMSE in Panel B for various ES forecasting models. See Table 1 for a description of the simulation setup. The results are based on simulations with leverage effects ($\theta = 0.5$). Bolded values indicate the model with the lowest error.

Table 5. Ranking of 1% VaR forecasting models based on the MCS procedure using quantile scores

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	16	1	3	7	2	1	1	1	8	4.0	1
GJR-GARCH-skew-t	15	6	12	4	7	8	3	4	8	7.4	2
GARCH-skew-t	20	3	2	2	9	6	16	6	8	8.0	3
GJR-GARCH-t	7	7	11	1	20	2	2	15	8	8.1	4
AR-GJR-GARCH-skew-t	1	5	7	3	10	18	12	9	8	8.1	5
AR-GARCH-skew-t	13	9	1	8	11	4	15	8	8	8.6	6
AR-GJR-GARCH-t	2	2	10	6	19	7	11	21	8	9.8	7
AR-$\text{AL}_{\text{Mult.}}$-AS	4	10	16	20	4	12	14	2	8	10.2	8
QAR-QbSD-gAS	6	4	22	5	1	14	28	3	8	10.4	9
$\text{AL}_{\text{Mult.}}$-AS	8	17	14	21	3	13	4	5	8	10.6	10
AR-AL_{AR}-AS	18	8	13	14	5	11	9	11	8	11.1	11
AL_{AR}-AS	14	18	15	19	6	10	5	7	8	11.8	12
GARCH-t	5	13	8	12	22	3	18	14	8	11.9	13
AR-GARCH-t	3	11	9	9	21	20	13	22	8	13.5	14
$\text{AL}_{\text{Mult.}}$-SAV	12	19	5	16	16	19	6	20	8	14.1	15
QAR-QbSD-gSAV	27	21	25	10	8	5	7	10	8	14.1	16
AR-$\text{AL}_{\text{Mult.}}$-SAV	9	14	6	17	18	22	10	19	8	14.4	17
EGARCH	21	15	18	11	14	16	19	12	8	15.8	18
AL_{AR}-SAV	22	24	4	18	13	17	17	17	8	16.5	19
AR-AL_{AR}-SAV	10	12	17	24	15	21	20	18	8	17.1	20
QbSD-gSAV		28	26	15	12	9	8	13	7	17.4	21
AR-EGARCH	19	16	19	13	17	26	21	16	8	18.4	22
GAS	11	22	20	27	23	27	24	23	8	22.1	23
AR-GAS	17	20	21		24		25	24	6	23.4	24
GARCH-normal	24	23	24	23	25	23	23		7	24.1	25
GJR-GARCH-normal	23	25	23	22		24	22		6	24.4	26
AR-GJR-GARCH-normal	25	26	28	25		15	26		6	25.1	27
AR-GARCH-normal	26	27	27	26		25	27		6	26.8	28

Notes: A model that is excluded from $\widehat{\mathcal{M}}^*_{90\%}$ for a given stock index is left blank in the table. The remaining models in $\widehat{\mathcal{M}}^*_{90\%}$ are ranked based on their quantile score t-statistics, with lower values indicating better performance. The column "#" represents the number of stock indices (out of 8) for which the model is included in $\widehat{\mathcal{M}}^*_{90\%}$. The "Avg. rank" column reports the average rank across all stock indices, where models eliminated from $\widehat{\mathcal{M}}^*_{90\%}$ were assigned a rank of 28. The "Final rank" column orders models from best to worst based on their average rank, summarizing their relative forecasting performance across stock indices. The rolling-window size is $R = 1250$.

Table 6. Ranking of 2.5% VaR forecasting models based on the MCS procedure using quantile scores

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	1	1	16	1	1	1	3	8	3.2	1
$\text{AL}_{\text{Mult.}}$-AS	9	3	2	9	2	2	2	2	8	3.9	2
AR-$\text{AL}_{\text{Mult.}}$-AS	6	4	6	8	3	3	3	1	8	4.2	3
AL_{AR}-AS	7	5	4	4	8	4	8	5	8	5.6	4
GJR-GARCH-skew-t	4	8	5	6	12	7	7	8	8	7.1	5
QAR-QbSD-gAS	1	2	23	2	6	5	15	4	8	7.2	6
GARCH-skew-t	3	6	7	1	10	8	19	7	8	7.6	7
EGARCH	10	7	8	12	5	11	12	6	8	8.9	8
AR-AL_{AR}-AS	12	12	3	11	4	10	26	13	8	11.4	9
GJR-GARCH-t	13	9	10	13	15	16	10	11	8	12.1	10
$\text{AL}_{\text{Mult.}}$-SAV	18	16	17	5	11	20	4	9	8	12.5	11
AR-GJR-GARCH-skew-t	5	13	11	19	14	9	17	15	8	12.9	12
AR-EGARCH	16	11	22	15	7	12	21	10	8	14.2	13
GARCH-t	11	10	12	26	16	17	11	12	8	14.4	14
AR-GARCH-skew-t	8	19	9	22	17	6	20	14	8	14.4	15
AL_{AR}-SAV	25	17	14	7	23	21	5	19	8	16.4	16
AR-$\text{AL}_{\text{Mult.}}$-SAV	14	20	19	3	19	22	6		7	16.4	17
GJR-GARCH-normal	21	18	13	14	22	23	13	18	8	17.8	18
QAR-QbSD-gSAV	27	22	25	10	13	18	16	16	8	18.4	19
GARCH-normal	22	15	20	17	18	25	14	17	8	18.5	20
AR-GJR-GARCH-t	15	14	15	20	20	15	22		7	18.6	21
QbSD-gSAV		26	18	9	13	9			5	19.9	22
AR-AL_{AR}-SAV	20	21	18	21	24	19	18		7	21.1	23
AR-GARCH-t	17	24	16	25	21	26	23		7	22.5	24
AR-GJR-GARCH-normal	19	25	21	24		14	24		6	22.9	25
AR-GARCH-normal	23	26	24	23	25	24	25		7	24.8	26
GAS	24	23							2	26.9	27
AR-GAS	26	27							2	27.6	28

Notes: For explanations, see notes of Table 5.

Table 7. Ranking of 5% VaR forecasting models based on the MCS procedure using quantile scores

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	1	1	1	12	1	1	1	3	8	2.6	1
EGARCH	3	3	2	2	2	4	6	5	8	3.4	2
$AL_{Mult.}$-AS	4	19	3	4	5	7	3	1	8	5.8	3
QAR-QbSD-gAS	2	2	6	14	4	2	12	4	8	5.8	4
AR-EGARCH	11	4	14	7	3	5	7	12	8	7.9	5
AR-$AL_{Mult.}$-AS	18	11	11	9	8	3	4	2	8	8.2	6
GJR-GARCH-skew-t	10	8	4	6	12	10	13	8	8	8.9	7
GARCH-skew-t	9	5	5	10	10	14	15	9	8	9.6	8
AL_{AR}-AS	20	18	15	1	7	6	5	7	8	9.9	9
GJR-GARCH-normal	7	10	10	3	21	11	10	10	8	10.2	10
AL_{AR}-AS	17	20	16	17	6	8	2	6	8	11.5	11
GJR-GARCH-t	6	6	12	22	9	16	8	14	8	11.6	12
GARCH-t	8	7	9	25	11	17	9	13	8	12.4	13
GARCH-normal	12	9	13	16	16	13	11	11	8	12.6	14
AR-GJR-GARCH-skew-t	5	15	7	21	18	18	16		7	16.0	15
AR-GARCH-skew-t	13	16	8	24	17	15	22		7	17.9	16
AR-GJR-GARCH-t	15	12	17	23	14	20	18		7	18.4	17
AR-GARCH-normal	16	17	18	18	20	12	21		7	18.8	18
AR-GJR-GARCH-normal	19	13	20	20	24	9	17		7	18.8	19
AR-GARCH-t	14	14	19	26	15	22	20		7	19.8	20
QbSD-gSAV	25	21	22	19	13	25	14	19	8	19.8	21
$AL_{Mult.}$-SAV	26		26	8	22	26	19	17	7	21.5	22
QAR-QbSD-gSAV	22		24	15	19	21		18	6	21.9	23
AL_{AR}-SAV	27		27	11	23	23		15	6	22.8	24
AR-AL_{AR}-SAV		21	21	5		19			3	23.1	25
AR-$AL_{Mult.}$-SAV	24		23	13		24		20	5	24.5	26
AR-GAS	23		25			27		16	4	25.4	27
GAS	21		28	27				3	3	27.0	28

Notes: For explanations, see notes of Table 5.

Table 8. Ranking of 1% VaR and ES forecasting models based on the MCS procedure using AL log scores

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
AR-GJR-GARCH-skew-t	1	5	3	1	3	3	8	11	8	4.4	1
QbSD-gAS	9	4	6	8	6	1	1	1	8	4.5	2
GARCH-skew-t	3	2	1	3	4	5	14	10	8	5.2	3
GJR-GARCH-skew-t	12	3	4	4	5	6	2	9	8	5.6	4
QAR-QbSD-gAS	4	1	22	7	1	12	4	4	8	6.9	5
AR-AL$_{\text{AR}}$-AS	5	8	11	11	12	7	7	5	8	8.2	6
AR-GARCH-skew-t	2	16	2	10	8	2	16	12	8	8.5	7
AR-AL$_{\text{Mult.}}$-AS	6	6	12	16	9	11	12	2	8	9.2	8
QAR-QbSD-gSAV	21	15	18	2	2	4	6	6	8	9.2	9
AL$_{\text{Mult.}}$-AS	7	18	10	17	10	14	5	3	8	10.5	10
AL$_{\text{AR}}$-AS	11	14	13	12	7	13	9	8	8	10.9	11
GJR-GARCH-t	17	9	15	6	21	8	3	18	8	12.1	12
AR-AL$_{\text{Mult.}}$-SAV	10	10	5	19	15	20	15	14	8	13.5	13
AR-GJR-GARCH-t	14	7	16	5	19	10	10		7	13.6	14
AR-AL$_{\text{AR}}$-SAV	8	11	7	20	13	16	22	15	8	14.0	15
AL$_{\text{Mult.}}$-SAV	13	17	8	18	16	19	13	13	8	14.6	16
GARCH-t	18	12	14	14	22	9	18	17	8	15.5	17
AL$_{\text{AR}}$-SAV	16	19	9	15	14	17	20	16	8	15.8	18
QbSD-gSAV	24	20	19	13	11	15	17	7	8	15.8	19
AR-GARCH-t	15	13	17	9	20	18	11		7	16.4	20
GAS	19		21		17	21	19	20	6	21.6	21
EGARCH	23	23	20	21	23	22		19	7	22.4	22
AR-GAS	20	21	24		18		21		5	23.5	23
AR-EGARCH	22	22	23	22	24				5	24.6	24
GARCH-normal									0	28.0	25
GJR-GARCH-normal									0	28.0	26
AR-GARCH-normal									0	28.0	27
AR-GJR-GARCH-normal									0	28.0	28

Notes: A model that is excluded from $\widehat{\mathcal{M}}^*_{90\%}$ for a given stock index is left blank in the table. The remaining models in $\widehat{\mathcal{M}}^*_{90\%}$ are ranked based on their AL log score t-statistics, with lower values indicating better performance. The column "#" represents the number of stock indices (out of 8) for which the model is included in $\widehat{\mathcal{M}}^*_{90\%}$. The "Avg. rank" column reports the average rank across all stock indices, where models eliminated from $\widehat{\mathcal{M}}^*_{90\%}$ were assigned a rank of 28. The "Final rank" column orders models from best to worst based on their average rank, summarizing their relative forecasting performance across stock indices. The rolling-window size is $R = 1250$.

Table 9. Ranking of 2.5% VaR and ES forecasting models based on the MCS procedure using AL log scores

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	5	2	3	12	1	1	1	1	8	3.2	1
AR-AL$_{\text{Mult.}}$-AS	7	8	8	4	4	2	3	4	8	5.0	2
AL$_{\text{Mult.}}$-AS	11	7	6	8	2	3	2	2	8	5.1	3
QAR-QbSD-gAS	1	1	18	2	3	8	4	5	8	5.2	4
GJR-GARCH-skew-t	4	6	1	3	9	7	6		7	8.0	5
AL$_{\text{AR}}$-AS	9	3	5	9		4	5	3	7	8.2	6
GARCH-skew-t	2	5	4	5	8	6	14		7	9.0	7
AR-AL$_{\text{AR}}$-AS	8	4	2	15	15	10	17	6	8	9.6	8
AR-GJR-GARCH-skew-t	3	9	9	11	11	9	9		7	11.1	9
QAR-QbSD-gSAV	14	13	14	1	5	11	8		7	11.8	10
AR-GARCH-skew-t	6	12	7	13	12	5	15		7	12.2	11
AL$_{\text{Mult.}}$-SAV	13	14	12	10	6	17	10		7	13.8	12
AR-AL$_{\text{Mult.}}$-SAV	10	18	11	6	10	15	12		7	13.8	13
AL$_{\text{AR}}$-SAV	17	15	13	7	13	14	11		7	14.8	14
AR-AL$_{\text{AR}}$-SAV	12	17	10	14	14	12	13		7	15.0	15
GJR-GARCH-t	16	10	15	16	16	18	16		7	16.9	16
QbSD-gSAV		22	22	18	7	13	7		6	18.1	17
AR-GJR-GARCH-t	18	11	19	17	19	16	19		7	18.4	18
GARCH-t	15	16	16	22	17	19	18		7	18.9	19
EGARCH	19	19	17	19	18	21			6	21.1	20
AR-GARCH-t	21	20	20	20		20	20		6	22.1	21
AR-EGARCH	20	21	21	21	20	22			6	22.6	22
GARCH-normal									0	28.0	23
GJR-GARCH-normal									0	28.0	24
AR-GARCH-normal									0	28.0	25
AR-GJR-GARCH-normal									0	28.0	26
GAS									0	28.0	27
AR-GAS									0	28.0	28

Notes: For explanations, see notes of Table 8.

Table 10. Ranking of 5% VaR and ES forecasting models based on the MCS procedure using AL log scores

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	1	1	5	1	1	1	2	8	1.8	1
AL_{AR}-AS	3	4	2	13	4	2	2	3	8	4.1	2
$AL_{Mult.}$-AS	4	6	5	3	3	5	4	4	8	4.2	3
QAR-QbSD-gAS	1	2	14	4	2	4	6	1	8	4.2	4
AR-AL_{AR}-AS	10	9	7	1	5	6	5	5	8	6.0	5
AR-$AL_{Mult.}$-AS	7	15	11	2	11	3	3	6	8	7.2	6
GARCH-skew-t	6	5	4	10	7	8		9	7	9.6	7
GJR-GARCH-skew-t	9	7	3	8	9	7		8	7	9.9	8
EGARCH	15	3	9	9	10	12		7	7	11.6	9
AR-GJR-GARCH-skew-t	5	8	6	15	12	11			6	14.1	10
QAR-QbSD-gSAV	11	16	10	6	6	9			6	14.2	11
AR-GARCH-skew-t	8	10	8	16	13	10			6	15.1	12
GJR-GARCH-t	12	12	13	18	17	16			6	18.0	13
AR-EGARCH	20	11	16	14	14	14			6	18.1	14
AR-AL_{AR}-SAV	16	18	18	11	22	13			6	19.2	15
GARCH-t	13	13	12	26	18	17			6	19.4	16
AL_{AR}-SAV	21	22	24	17	15	18		11	7	19.5	17
AR-$AL_{Mult.}$-SAV	17		15	7	21	15			5	19.9	18
AR-GJR-GARCH-t	14	14	22	19	20	19			6	20.5	19
$AL_{Mult.}$-SAV		20	11	16	22		10	5	5	20.5	20
AR-GARCH-t	18	17	23	22	19	20			6	21.9	21
QbSD-gSAV	22	20		20	8	21			5	21.9	22
GJR-GARCH-normal	19	21	19	21		25			5	23.6	23
AR-GJR-GARCH-normal		19		24		23			3	25.8	24
AR-GAS	24		17						2	26.1	25
GAS	23		21						2	26.5	26
GARCH-normal			22		26				2	27.1	27
AR-GARCH-normal			25	25		25			2	27.1	28

Notes: For explanations, see notes of Table 8.

Figure 1: Time series plots of the daily log returns for the S&P 500, DJIA, NASDAQ, and EURO STOXX 50 indices over the sample period. Each plot represents the percentage daily returns computed from adjusted closing prices.

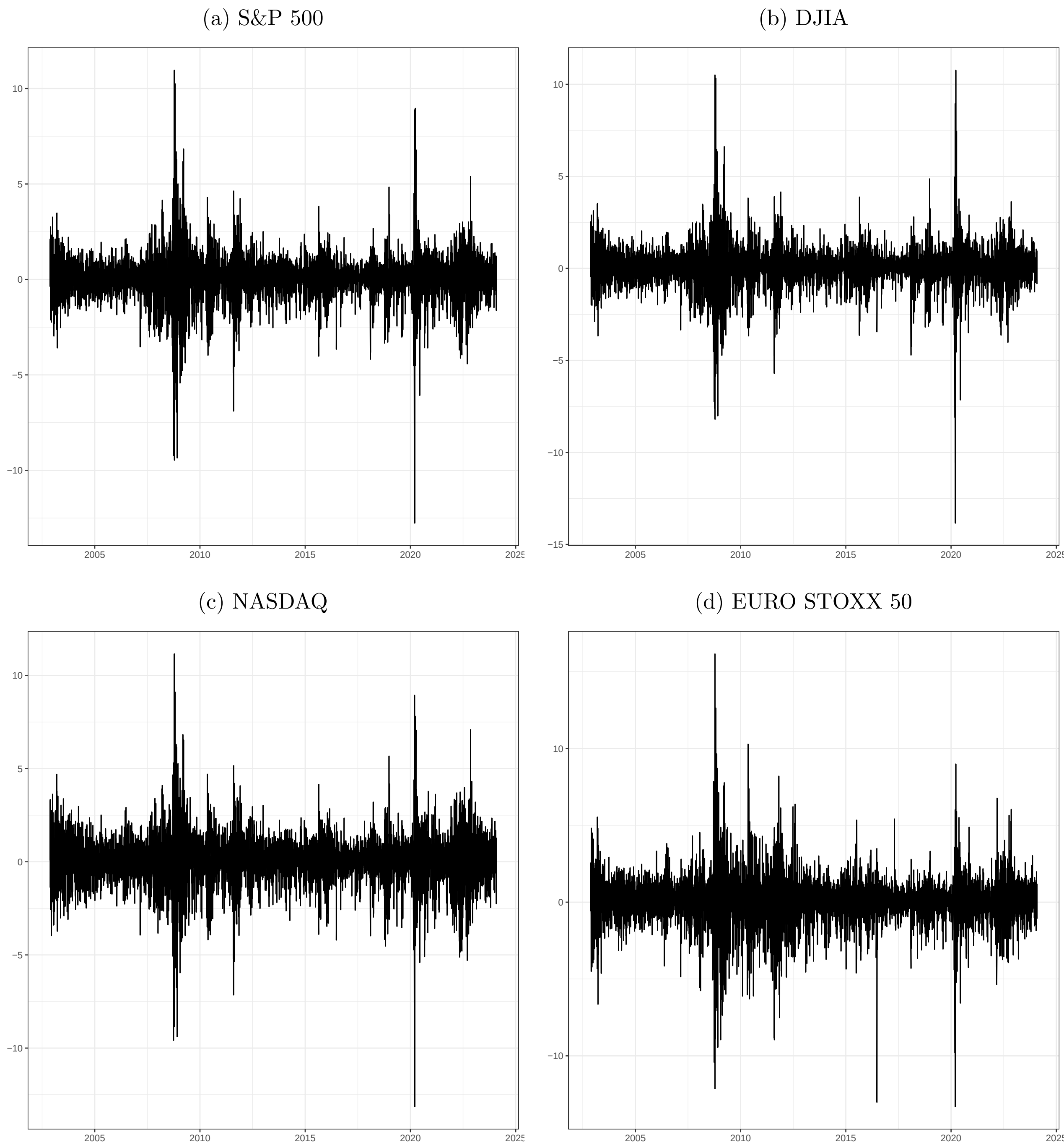
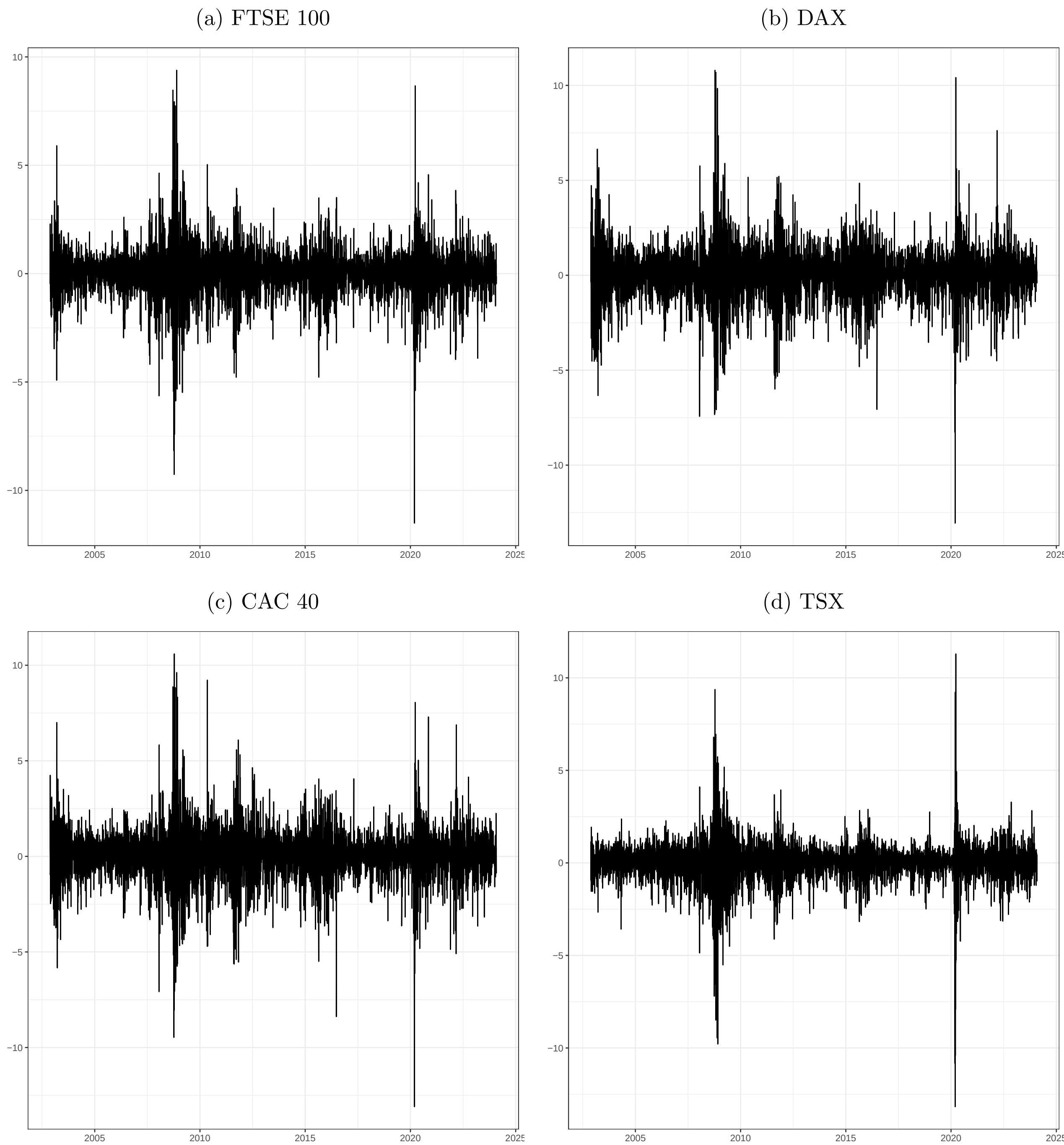


Figure 2: Time series plots of the daily log returns for the FTSE 100, DAX, CAC 40, and TSX indices over the sample period. Each plot displays percentage daily returns based on adjusted closing prices.



Supplementary material for:

Quantile-based modeling of scale dynamics in financial returns for Value-at-Risk and Expected Shortfall forecasting

Xiaochun Liu and Richard Luger

Section A

Tables A1–A12 present comparisons of the mean absolute error (MAE) and root mean squared error (RMSE) for the QbSD forecasting approach, examining the impact of the scale-defining quantile level p on the computation of VaR and ES. Specifically, we consider forecasts based on: (i) individual values of $p \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$; (ii) the mean over these p values; and (iii) the median over them.

The results are obtained from 1,000 replications for each configuration of the asymmetric power ARCH model with innovations drawn from the skewed t-distribution, as specified in the main text. The lowest MAE and RMSE values in each table are highlighted in bold.

Overall, the results in Tables A1–A12 suggest that combining forecasts over multiple quantile levels p—either by taking the mean or, typically to a slightly lesser degree, the median—tends to yield lower forecast errors than relying on a single p in isolation. This advantage of averaging is most noticeable when T is relatively small (e.g., $T = 250$; see especially Tables A1–A4), where individual-p estimates exhibit higher variability. As T increases to 1250 (Tables A5–A8) or 2500 (Tables A9–A12), the performance gap narrows, yet the multi-p strategies remain competitive or superior under most DGP configurations.

Although occasional scenarios arise where an individual p may match or slightly outperform the combined forecasts (e.g., $p = 0.10$ in Table A7), no single p exhibits consistent dominance across the wide range of parameter settings (leverage θ, tail thickness v, skewness λ) or sample sizes T. Consequently, our findings recommend the "mean over p" method as a robust, practically simple strategy for balancing estimation accuracy and stability in quantile-based VaR and ES forecasting.

Table A1. VaR forecasting results with the QbSD approach when $\theta = 0$ and $T = 250$

	$v = 20$, $\lambda = 0$			$v = 20$, $\lambda = -0.5$			$v = 5$, $\lambda = 0$			$v = 5$, $\lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.243	0.181	0.146	0.304	0.222	0.173	0.328	0.214	0.158	0.436	0.269	0.189
$p = 0.10$	0.235	0.172	0.138	0.281	0.204	0.157	0.318	0.206	0.148	0.402	0.251	0.173
$p = 0.15$	0.228	0.170	0.139	0.287	0.206	0.161	0.318	0.209	0.149	0.407	0.252	0.177
$p = 0.20$	0.237	0.181	0.149	0.297	0.220	0.171	0.319	0.212	0.158	0.422	0.260	0.185
$p = 0.25$	0.246	0.190	0.152	0.302	0.225	0.176	0.325	0.220	0.159	0.427	0.265	0.186
Mean over p	**0.215**	**0.159**	**0.130**	**0.272**	**0.197**	**0.154**	**0.298**	**0.193**	**0.140**	**0.394**	**0.241**	**0.169**
Median over p	0.222	0.164	0.134	0.277	0.202	0.157	0.308	0.198	0.143	0.401	0.247	0.171
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.265	0.196	0.160	0.329	0.233	0.182	0.349	0.228	0.174	0.460	0.280	0.199
$p = 0.10$	0.259	0.190	0.156	0.307	0.220	0.173	0.340	0.227	0.168	0.421	0.268	0.188
$p = 0.15$	0.266	0.203	0.166	0.314	0.230	0.181	0.344	0.228	0.171	0.423	0.269	0.189
$p = 0.20$	0.275	0.213	0.175	0.326	0.244	0.196	0.354	0.240	0.181	0.440	0.283	0.207
$p = 0.25$	0.291	0.228	0.183	0.342	0.257	0.205	0.371	0.256	0.190	0.450	0.291	0.211
Mean over p	**0.238**	**0.178**	**0.146**	**0.288**	**0.211**	**0.168**	**0.315**	**0.208**	**0.155**	**0.402**	**0.251**	**0.180**
Median over p	0.247	0.185	0.152	0.297	0.219	0.172	0.324	0.214	0.161	0.412	0.258	0.184
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.308	0.232	0.188	0.388	0.291	0.228	0.422	0.287	0.214	0.567	0.370	0.267
$p = 0.10$	0.305	0.228	0.184	0.360	0.268	0.206	0.417	0.282	0.204	0.518	0.341	0.232
$p = 0.15$	0.290	0.222	0.180	0.363	0.269	0.212	0.410	0.280	0.203	0.532	0.344	0.240
$p = 0.20$	0.306	0.238	0.194	0.381	0.287	0.225	0.415	0.287	0.212	0.549	0.357	0.255
$p = 0.25$	0.318	0.253	0.202	0.390	0.299	0.235	0.422	0.301	0.215	0.565	0.368	0.262
Mean over p	**0.276**	**0.208**	**0.169**	**0.346**	**0.257**	**0.201**	**0.386**	**0.260**	**0.188**	**0.513**	**0.330**	**0.231**
Median over p	0.283	0.214	0.175	0.353	0.265	0.205	0.394	0.267	0.192	0.520	0.336	0.232
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.338	0.256	0.208	0.414	0.302	0.240	0.454	0.308	0.240	0.587	0.374	0.274
$p = 0.10$	0.339	0.252	0.207	0.390	0.286	0.223	0.449	0.305	0.229	0.545	0.366	0.251
$p = 0.15$	0.348	0.266	0.217	0.405	0.300	0.238	0.454	0.308	0.230	0.565	0.363	0.253
$p = 0.20$	0.363	0.287	0.232	0.426	0.325	0.259	0.477	0.329	0.244	0.597	0.390	0.283
$p = 0.25$	0.394	0.317	0.250	0.453	0.350	0.278	0.512	0.367	0.264	0.606	0.400	0.289
Mean over p	**0.311**	**0.236**	**0.191**	**0.371**	**0.273**	**0.217**	**0.415**	**0.280**	**0.207**	**0.529**	**0.338**	**0.239**
Median over p	0.324	0.246	0.201	0.381	0.285	0.222	0.426	0.289	0.217	0.544	0.349	0.244

Table A2. ES forecasting results with the QbSD approach when $\theta = 0$ and $T = 250$

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.325	0.238	0.191	0.424	0.299	0.232	0.551	0.344	0.246	0.762	0.458	0.315
$p = 0.10$	0.311	0.226	0.180	0.395	0.275	0.212	0.537	0.332	0.234	0.732	0.429	0.290
$p = 0.15$	0.303	0.221	0.176	0.395	0.280	0.215	0.532	0.332	0.236	0.729	0.431	0.293
$p = 0.20$	0.311	0.229	0.185	0.409	0.291	0.225	0.530	0.331	0.237	0.739	0.445	0.305
$p = 0.25$	0.319	0.241	0.195	0.413	0.297	0.234	0.538	0.341	0.246	0.744	0.447	0.306
Mean over p	**0.293**	**0.211**	**0.167**	**0.387**	**0.269**	**0.206**	**0.519**	**0.317**	**0.221**	**0.724**	**0.424**	**0.285**
Median over p	0.302	0.217	0.172	0.393	0.275	0.210	0.526	0.323	0.227	0.728	0.429	0.289
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.353	0.260	0.210	0.457	0.327	0.254	0.578	0.368	0.266	0.805	0.494	0.343
$p = 0.10$	0.337	0.252	0.203	0.424	0.301	0.234	0.549	0.350	0.255	0.749	0.449	0.310
$p = 0.15$	0.340	0.259	0.212	0.426	0.307	0.242	0.557	0.357	0.260	0.744	0.450	0.312
$p = 0.20$	0.346	0.268	0.221	0.431	0.319	0.255	0.555	0.362	0.270	0.748	0.462	0.327
$p = 0.25$	0.355	0.281	0.234	0.447	0.332	0.268	0.571	0.378	0.282	0.770	0.476	0.336
Mean over p	**0.311**	**0.233**	**0.189**	**0.403**	**0.286**	**0.226**	**0.528**	**0.331**	**0.238**	**0.734**	**0.437**	**0.300**
Median over p	0.321	0.241	0.197	0.410	0.294	0.231	0.539	0.338	0.246	0.741	0.444	0.306
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.406	0.302	0.242	0.526	0.383	0.302	0.683	0.440	0.321	0.943	0.591	0.420
$p = 0.10$	0.395	0.294	0.237	0.488	0.352	0.274	0.665	0.426	0.311	0.885	0.539	0.375
$p = 0.15$	0.378	0.282	0.227	0.489	0.352	0.276	0.656	0.420	0.305	0.893	0.544	0.379
$p = 0.20$	0.390	0.296	0.242	0.507	0.369	0.291	0.656	0.421	0.310	0.913	0.562	0.395
$p = 0.25$	0.408	0.312	0.256	0.518	0.380	0.304	0.672	0.436	0.321	0.930	0.577	0.408
Mean over p	**0.366**	**0.270**	**0.216**	**0.477**	**0.340**	**0.265**	**0.637**	**0.400**	**0.288**	**0.881**	**0.534**	**0.370**
Median over p	0.376	0.277	0.223	0.485	0.347	0.270	0.647	0.407	0.294	0.886	0.538	0.374
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.441	0.332	0.270	0.561	0.410	0.323	0.719	0.471	0.350	0.981	0.617	0.437
$p = 0.10$	0.432	0.328	0.266	0.527	0.381	0.299	0.705	0.458	0.339	0.923	0.568	0.402
$p = 0.15$	0.440	0.339	0.277	0.534	0.391	0.312	0.711	0.464	0.343	0.934	0.574	0.405
$p = 0.20$	0.448	0.352	0.293	0.550	0.413	0.335	0.717	0.480	0.360	0.945	0.596	0.429
$p = 0.25$	0.478	0.384	0.322	0.576	0.439	0.359	0.745	0.513	0.390	0.981	0.621	0.447
Mean over p	**0.402**	**0.305**	**0.248**	**0.504**	**0.364**	**0.287**	**0.669**	**0.429**	**0.314**	**0.905**	**0.551**	**0.385**
Median over p	0.416	0.316	0.259	0.512	0.373	0.296	0.684	0.439	0.324	0.915	0.561	0.394

Table A3. VaR forecasting results with the QbSD approach when $\theta = 0.5$ and $T = 250$

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p=0.05$	0.327	0.252	0.206	0.415	0.314	0.245	0.401	0.277	0.206	0.553	0.355	0.259
$p=0.10$	0.316	0.241	0.195	0.397	0.295	0.231	0.380	0.257	0.190	0.514	0.333	0.237
$p=0.15$	0.316	0.243	0.197	0.404	0.301	0.235	0.381	0.257	0.192	0.528	0.338	0.238
$p=0.20$	0.324	0.247	0.202	0.423	0.316	0.248	0.388	0.266	0.195	0.536	0.347	0.245
$p=0.25$	0.333	0.255	0.205	0.442	0.327	0.258	0.402	0.273	0.202	0.574	0.367	0.264
Mean over p	**0.303**	**0.231**	**0.187**	**0.390**	**0.291**	**0.228**	**0.369**	**0.249**	**0.184**	**0.510**	**0.326**	**0.233**
Median over p	0.307	0.232	0.190	0.395	0.294	0.231	0.373	0.250	0.186	0.515	0.330	0.235
QbSD-gAS model												
Individual values of p												
$p=0.05$	0.283	0.213	0.172	0.388	0.276	0.222	0.380	0.252	0.193	0.546	0.344	0.250
$p=0.10$	0.272	0.207	0.167	0.354	0.257	0.203	0.357	0.238	0.177	0.505	0.318	0.226
$p=0.15$	0.288	0.220	0.178	0.369	0.269	0.215	0.365	0.245	0.184	0.510	0.317	0.233
$p=0.20$	0.304	0.233	0.189	0.388	0.288	0.230	0.379	0.258	0.195	0.525	0.336	0.244
$p=0.25$	0.327	0.253	0.205	0.416	0.315	0.252	0.399	0.270	0.201	0.557	0.361	0.262
Mean over p	**0.257**	**0.196**	**0.160**	**0.339**	**0.248**	**0.199**	**0.335**	**0.222**	**0.168**	**0.482**	**0.302**	**0.218**
Median over p	0.266	0.199	0.164	0.350	0.257	0.205	0.343	0.227	0.170	0.490	0.306	0.226
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p=0.05$	0.441	0.347	0.285	0.571	0.440	0.345	0.549	0.396	0.300	0.766	0.528	0.387
$p=0.10$	0.417	0.326	0.267	0.526	0.402	0.319	0.512	0.366	0.274	0.707	0.485	0.348
$p=0.15$	0.420	0.331	0.271	0.548	0.421	0.331	0.512	0.365	0.278	0.740	0.497	0.351
$p=0.20$	0.437	0.342	0.278	0.569	0.443	0.350	0.532	0.380	0.281	0.733	0.500	0.353
$p=0.25$	0.449	0.354	0.282	0.604	0.466	0.362	0.571	0.395	0.297	0.849	0.551	0.394
Mean over p	**0.401**	**0.314**	**0.255**	**0.522**	**0.403**	**0.318**	**0.500**	**0.353**	**0.265**	**0.707**	**0.477**	**0.339**
Median over p	0.406	0.317	0.262	0.534	0.409	0.322	0.502	0.351	0.268	0.711	0.487	0.343
QbSD-gAS model												
Individual values of p												
$p=0.05$	0.378	0.291	0.240	0.523	0.383	0.318	0.517	0.356	0.281	0.762	0.513	0.393
$p=0.10$	0.365	0.284	0.228	0.476	0.363	0.286	0.484	0.342	0.250	0.690	0.486	0.344
$p=0.15$	0.391	0.303	0.244	0.512	0.387	0.309	0.499	0.329	0.252	0.772	0.483	0.357
$p=0.20$	0.421	0.328	0.266	0.547	0.411	0.330	0.550	0.367	0.278	0.815	0.516	0.378
$p=0.25$	0.456	0.367	0.296	0.595	0.470	0.367	0.605	0.396	0.295	0.815	0.546	0.393
Mean over p	**0.340**	**0.262**	**0.214**	**0.454**	**0.342**	**0.276**	**0.455**	**0.302**	**0.229**	**0.675**	**0.443**	**0.322**
Median over p	0.356	0.270	0.225	0.479	0.363	0.290	0.468	0.310	0.236	0.702	0.459	0.342

Table A4. ES forecasting results with the QbSD approach when $\theta = 0.5$ and $T = 250$

	$v = 20, \lambda = 0$			$v = 20, \lambda = -0.5$			$v = 5, \lambda = 0$			$v = 5, \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.420	0.324	0.266	0.557	0.414	0.329	0.654	0.425	0.314	0.945	0.585	0.412
$p = 0.10$	0.401	0.309	0.254	0.527	0.392	0.311	0.619	0.398	0.291	0.888	0.539	0.379
$p = 0.15$	0.396	0.308	0.254	0.537	0.399	0.316	0.611	0.393	0.289	0.907	0.553	0.386
$p = 0.20$	0.401	0.314	0.259	0.549	0.412	0.329	0.625	0.403	0.298	0.913	0.563	0.397
$p = 0.25$	0.409	0.323	0.266	0.566	0.431	0.344	0.635	0.414	0.306	0.942	0.593	0.423
Mean over p	**0.385**	**0.298**	**0.244**	**0.524**	**0.387**	**0.307**	**0.608**	**0.388**	**0.284**	**0.894**	**0.540**	**0.376**
Median over p	0.391	0.301	0.246	0.529	0.390	0.308	0.613	0.392	0.286	0.902	0.544	0.378
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.382	0.281	0.227	0.544	0.386	0.301	0.629	0.401	0.292	0.948	0.591	0.416
$p = 0.10$	0.357	0.268	0.216	0.490	0.349	0.272	0.584	0.370	0.269	0.883	0.535	0.373
$p = 0.15$	0.366	0.279	0.227	0.507	0.363	0.285	0.590	0.376	0.276	0.888	0.538	0.372
$p = 0.20$	0.377	0.293	0.242	0.511	0.377	0.301	0.595	0.387	0.288	0.891	0.550	0.387
$p = 0.25$	0.402	0.316	0.264	0.538	0.407	0.331	0.613	0.406	0.304	0.913	0.572	0.414
Mean over p	**0.337**	**0.253**	**0.207**	**0.475**	**0.337**	**0.266**	**0.566**	**0.353**	**0.256**	**0.869**	**0.518**	**0.360**
Median over p	0.347	0.261	0.212	0.487	0.348	0.273	0.575	0.360	0.262	0.874	0.524	0.364
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.557	0.437	0.364	0.746	0.565	0.458	0.851	0.576	0.438	1.246	0.801	0.586
$p = 0.10$	0.522	0.408	0.340	0.694	0.521	0.421	0.795	0.529	0.400	1.147	0.728	0.530
$p = 0.15$	0.523	0.412	0.343	0.715	0.539	0.434	0.788	0.526	0.397	1.180	0.749	0.541
$p = 0.20$	0.540	0.428	0.356	0.732	0.559	0.453	0.812	0.545	0.412	1.207	0.762	0.549
$p = 0.25$	0.547	0.436	0.364	0.764	0.591	0.480	0.843	0.572	0.433	1.300	0.843	0.613
Mean over p	**0.506**	**0.396**	**0.328**	**0.689**	**0.520**	**0.419**	**0.782**	**0.519**	**0.389**	**1.167**	**0.734**	**0.527**
Median over p	0.512	0.400	0.333	0.701	0.528	0.425	0.790	0.522	0.391	1.174	0.740	0.532
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.489	0.372	0.305	0.705	0.517	0.414	0.813	0.537	0.403	1.270	0.812	0.593
$p = 0.10$	0.467	0.357	0.294	0.639	0.467	0.373	0.751	0.491	0.366	1.146	0.714	0.515
$p = 0.15$	0.482	0.374	0.310	0.676	0.499	0.401	0.770	0.498	0.368	1.185	0.745	0.536
$p = 0.20$	0.510	0.404	0.337	0.691	0.519	0.422	0.797	0.534	0.403	1.205	0.768	0.563
$p = 0.25$	0.547	0.442	0.375	0.744	0.579	0.478	0.837	0.577	0.439	1.267	0.817	0.600
Mean over p	**0.439**	**0.333**	**0.274**	**0.619**	**0.449**	**0.359**	**0.727**	**0.464**	**0.340**	**1.125**	**0.689**	**0.491**
Median over p	0.454	0.346	0.284	0.640	0.469	0.375	0.745	0.475	0.349	1.144	0.705	0.509

Table A5. VaR forecasting results with the QbSD approach when $\theta = 0$ and $T = 1250$

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.106	0.080	0.064	0.138	0.099	0.077	0.152	0.099	0.073	0.208	0.126	0.087
$p = 0.10$	0.106	0.078	0.061	0.135	0.094	0.072	0.148	0.092	0.065	0.201	0.117	0.079
$p = 0.15$	0.108	0.079	0.062	0.136	0.096	0.073	0.149	0.094	0.066	0.201	0.119	0.081
$p = 0.20$	0.109	0.081	0.064	0.139	0.097	0.075	0.149	0.094	0.067	0.201	0.119	0.082
$p = 0.25$	0.115	0.088	0.068	0.143	0.103	0.079	0.151	0.099	0.070	0.202	0.121	0.083
Mean over p	**0.101**	**0.075**	**0.059**	**0.132**	**0.092**	**0.071**	**0.142**	**0.089**	**0.063**	**0.196**	**0.115**	**0.079**
Median over p	0.103	0.076	0.060	0.133	0.093	0.072	0.144	0.090	0.064	0.198	0.116	**0.079**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.117	0.088	0.071	0.144	0.106	0.082	0.163	0.108	0.081	0.213	0.133	0.093
$p = 0.10$	0.116	0.086	0.069	0.142	0.102	0.077	0.156	0.101	0.073	0.207	0.127	0.086
$p = 0.15$	0.116	0.087	0.069	0.142	0.101	0.077	0.156	0.100	0.073	0.204	0.125	0.084
$p = 0.20$	0.123	0.092	0.072	0.147	0.103	0.080	0.160	0.104	0.075	0.205	0.123	0.086
$p = 0.25$	0.129	0.099	0.079	0.150	0.112	0.087	0.163	0.108	0.078	0.205	0.126	0.088
Mean over p	**0.109**	**0.081**	**0.064**	**0.136**	**0.096**	**0.073**	**0.148**	**0.094**	**0.068**	**0.197**	**0.118**	**0.081**
Median over p	0.110	0.082	0.065	0.137	0.097	**0.073**	0.150	0.096	0.069	0.199	0.120	0.082
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.139	0.107	0.085	0.182	0.133	0.103	0.204	0.134	0.099	0.275	0.169	0.119
$p = 0.10$	0.135	0.102	0.080	0.175	0.123	0.093	0.192	0.121	0.087	0.265	0.155	0.105
$p = 0.15$	0.137	0.103	0.081	0.175	0.125	0.094	0.194	0.124	0.088	0.264	0.158	0.106
$p = 0.20$	0.141	0.107	0.084	0.178	0.127	0.097	0.199	0.126	0.090	0.265	0.158	0.108
$p = 0.25$	0.151	0.115	0.090	0.184	0.133	0.102	0.201	0.130	0.092	0.264	0.159	0.109
Mean over p	**0.129**	**0.097**	**0.076**	**0.168**	**0.119**	**0.090**	**0.186**	**0.117**	**0.083**	**0.256**	**0.151**	**0.102**
Median over p	0.131	0.098	0.077	0.170	0.121	0.092	0.189	0.118	0.084	0.258	0.152	0.103
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.154	0.118	0.095	0.190	0.140	0.108	0.216	0.145	0.109	0.283	0.178	0.126
$p = 0.10$	0.150	0.114	0.090	0.185	0.134	0.101	0.205	0.133	0.097	0.271	0.168	0.114
$p = 0.15$	0.150	0.114	0.090	0.185	0.134	0.102	0.204	0.132	0.097	0.270	0.166	0.112
$p = 0.20$	0.159	0.121	0.096	0.190	0.138	0.105	0.213	0.140	0.102	0.269	0.165	0.113
$p = 0.25$	0.170	0.132	0.104	0.193	0.146	0.112	0.215	0.144	0.105	0.267	0.166	0.117
Mean over p	**0.139**	**0.106**	**0.083**	**0.174**	**0.126**	**0.095**	**0.193**	**0.124**	**0.090**	**0.257**	**0.156**	**0.106**
Median over p	0.141	0.107	0.084	0.175	0.127	**0.095**	0.196	0.126	0.091	0.258	0.157	0.107

Table A6. ES forecasting results with the QbSD approach when $\theta = 0$ and $T = 1250$

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute er ror												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.144	0.105	0.087	0.191	0.137	0.109	0.263	0.164	0.121	0.367	0.222	0.160
$p = 0.10$	0.139	0.102	0.085	0.184	0.131	0.105	0.254	0.155	0.114	0.357	0.213	0.151
$p = 0.15$	0.140	0.104	0.087	0.183	0.131	0.106	0.251	0.156	0.115	0.356	0.213	0.151
$p = 0.20$	0.142	0.106	0.087	0.187	0.133	0.107	0.253	0.157	0.115	0.356	0.213	0.150
$p = 0.25$	0.148	0.111	0.092	0.191	0.137	0.111	0.255	0.158	0.117	0.356	0.214	0.150
Mean over p	**0.136**	**0.100**	**0.082**	**0.181**	**0.128**	**0.103**	**0.248**	**0.151**	**0.111**	**0.353**	**0.210**	**0.148**
Median over p	0.137	0.101	0.083	0.183	0.129	0.104	0.250	0.152	0.112	0.355	0.211	**0.148**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.153	0.114	0.096	0.198	0.143	0.116	0.271	0.170	0.129	0.377	0.231	0.168
$p = 0.10$	0.149	0.113	0.094	0.194	0.140	0.113	0.261	0.163	0.123	0.366	0.221	0.160
$p = 0.15$	0.147	0.113	0.093	0.189	0.138	0.112	0.255	0.161	0.121	0.360	0.216	0.157
$p = 0.20$	0.153	0.118	0.098	0.194	0.142	0.115	0.260	0.165	0.125	0.357	0.215	0.155
$p = 0.25$	0.160	0.124	0.104	0.195	0.146	0.119	0.263	0.168	0.126	**0.354**	0.216	0.156
Mean over p	**0.140**	**0.106**	**0.088**	**0.185**	**0.133**	**0.107**	**0.251**	**0.155**	**0.116**	**0.354**	**0.211**	**0.152**
Median over p	0.142	0.107	0.089	0.186	**0.133**	0.108	0.253	**0.155**	0.117	0.356	0.212	0.153
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.184	0.138	0.115	0.247	0.180	0.146	0.335	0.215	0.160	0.469	0.289	0.209
$p = 0.10$	0.177	0.132	0.109	0.233	0.168	0.134	0.316	0.199	0.146	0.446	0.271	0.193
$p = 0.15$	0.179	0.133	0.111	0.234	0.170	0.137	0.319	0.202	0.149	0.444	0.272	0.194
$p = 0.20$	0.182	0.138	0.115	0.238	0.173	0.139	0.320	0.206	0.153	0.444	0.272	0.195
$p = 0.25$	0.191	0.146	0.121	0.242	0.177	0.143	0.321	0.204	0.152	0.442	0.271	0.193
Mean over p	**0.172**	**0.127**	**0.105**	**0.229**	**0.164**	**0.132**	**0.312**	**0.195**	**0.143**	**0.440**	**0.266**	**0.189**
Median over p	0.174	0.129	0.107	0.231	0.166	0.133	0.314	0.197	0.144	0.442	0.267	0.190
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.197	0.151	0.127	0.251	0.187	0.152	0.346	0.225	0.172	0.472	0.295	0.217
$p = 0.10$	0.191	0.146	0.122	0.246	0.180	0.145	0.329	0.211	0.159	0.456	0.281	0.204
$p = 0.15$	0.189	0.144	0.121	0.244	0.179	0.145	0.327	0.209	0.157	0.448	0.277	0.201
$p = 0.20$	0.198	0.154	0.128	0.248	0.184	0.150	0.333	0.218	0.164	0.447	0.276	0.201
$p = 0.25$	0.208	0.163	0.137	0.248	0.187	0.152	0.332	0.217	0.164	0.441	0.272	0.198
Mean over p	**0.180**	**0.136**	**0.113**	**0.233**	**0.170**	**0.138**	**0.318**	**0.201**	**0.150**	**0.439**	**0.267**	**0.193**
Median over p	0.181	0.137	0.115	0.234	0.171	**0.138**	0.320	0.202	0.151	0.441	0.268	0.194

Table A7. VaR forecasting results with the QbSD approach when $\theta = 0.5$ and $T = 1250$

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.226	0.180	0.147	0.286	0.218	0.171	0.252	0.180	0.138	0.339	0.224	0.162
$p = 0.10$	0.225	**0.178**	0.145	0.286	0.218	0.170	0.250	0.173	0.132	0.339	0.222	0.157
$p = 0.15$	0.230	0.181	0.148	0.289	0.220	0.172	0.252	0.176	0.135	0.342	0.225	0.160
$p = 0.20$	0.231	0.183	0.149	0.291	0.223	0.174	0.250	0.177	0.134	0.339	0.225	0.162
$p = 0.25$	0.235	0.185	0.151	0.296	0.226	0.178	0.255	0.180	0.139	0.342	0.226	0.163
Mean over p	**0.225**	**0.178**	**0.145**	**0.284**	**0.217**	**0.169**	**0.247**	**0.173**	**0.132**	**0.333**	**0.219**	**0.156**
Median over p	0.226	0.179	0.146	**0.284**	**0.217**	**0.169**	0.248	0.174	0.133	0.335	0.220	0.157
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.128	0.099	0.080	0.177	0.133	0.103	0.175	0.119	0.087	0.258	0.164	0.114
$p = 0.10$	0.126	0.097	0.077	0.171	0.129	0.098	0.170	0.113	0.082	0.248	0.159	0.108
$p = 0.15$	0.130	0.098	0.079	0.177	0.131	0.101	0.170	0.111	0.082	0.251	0.163	0.112
$p = 0.20$	0.134	0.103	0.081	0.180	0.134	0.103	0.176	0.116	0.084	0.252	0.163	0.112
$p = 0.25$	0.140	0.109	0.087	0.186	0.142	0.110	0.176	0.118	0.086	0.254	0.165	0.114
Mean over p	**0.120**	**0.092**	**0.072**	**0.164**	**0.124**	**0.094**	**0.161**	**0.106**	**0.076**	**0.239**	**0.152**	**0.104**
Median over p	0.121	0.093	0.074	0.167	0.125	0.095	0.164	0.107	0.078	0.242	0.155	0.106
Panel B: Root mean square d error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	**0.297**	**0.236**	**0.193**	0.383	0.294	0.228	0.345	0.246	0.188	0.461	0.312	0.220
$p = 0.10$	0.299	0.238	**0.193**	0.385	0.295	0.228	0.333	0.236	0.178	0.465	0.309	0.216
$p = 0.15$	0.305	0.243	0.198	0.392	0.300	0.232	0.340	0.241	0.183	0.474	0.316	0.219
$p = 0.20$	0.307	0.245	0.200	0.395	0.305	0.236	0.340	0.244	0.185	0.477	0.320	0.224
$p = 0.25$	0.309	0.247	0.202	0.401	0.308	0.240	0.345	0.246	0.187	0.483	0.323	0.228
Mean over p	**0.297**	**0.236**	**0.193**	**0.382**	**0.293**	**0.227**	**0.331**	**0.235**	**0.178**	**0.456**	**0.304**	**0.212**
Median over p	0.299	0.238	0.194	0.383	0.294	**0.227**	0.333	0.236	0.179	0.458	0.305	**0.212**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.174	0.140	0.110	0.248	0.190	0.145	0.245	0.170	0.125	0.371	0.239	0.166
$p = 0.10$	0.167	0.132	0.102	0.235	0.180	0.133	0.230	0.157	0.112	0.347	0.228	0.151
$p = 0.15$	0.175	0.135	0.107	0.244	0.183	0.139	0.228	0.155	0.111	0.366	0.240	0.162
$p = 0.20$	0.181	0.144	0.113	0.248	0.188	0.143	0.238	0.162	0.116	0.361	0.242	0.163
$p = 0.25$	0.187	0.152	0.120	0.259	0.203	0.155	0.237	0.163	0.119	0.371	0.249	0.172
Mean over p	**0.159**	**0.126**	**0.098**	**0.225**	**0.172**	**0.129**	**0.215**	**0.146**	**0.103**	**0.334**	**0.218**	**0.145**
Median over p	0.160	0.127	0.099	0.228	0.173	0.130	0.218	0.147	0.104	0.345	0.223	0.151

Table A8. ES forecasting results with the QbSD approach when $\theta = 0.5$ and $T = 1250$

	$v=20,\ \lambda=0$			$v=20,\ \lambda=-0.5$			$v=5,\ \lambda=0$			$v=5,\ \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.272	0.223	0.190	0.360	0.284	0.237	0.377	0.266	0.206	0.539	0.357	0.268
$p = 0.10$	**0.270**	**0.222**	0.188	0.357	0.283	0.236	0.368	0.257	0.198	0.534	0.354	0.265
$p = 0.15$	0.273	0.225	0.191	0.358	0.285	0.237	0.368	0.259	0.200	0.530	0.353	0.267
$p = 0.20$	0.276	0.227	0.193	0.362	0.288	0.240	0.368	0.259	0.201	0.530	0.353	0.267
$p = 0.25$	0.282	0.231	0.196	0.368	0.292	0.244	0.374	0.265	0.205	0.531	0.355	0.268
Mean over p	**0.270**	**0.222**	**0.189**	**0.355**	**0.282**	0.235	**0.365**	**0.256**	**0.198**	**0.529**	**0.349**	**0.262**
Median over p	0.271	0.223	0.190	**0.355**	**0.282**	**0.234**	0.367	0.257	0.199	0.527	**0.349**	**0.262**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.173	0.129	0.107	0.246	0.180	0.144	0.299	0.190	0.142	0.455	0.281	0.203
$p = 0.10$	0.164	0.124	0.104	0.231	0.171	0.138	0.282	0.179	0.134	0.432	0.265	0.193
$p = 0.15$	0.163	0.125	0.104	0.232	0.173	0.141	0.277	0.176	0.132	0.431	0.267	0.196
$p = 0.20$	0.170	0.131	0.108	0.236	0.177	0.145	0.281	0.182	0.137	0.427	0.265	0.196
$p = 0.25$	0.175	0.136	0.113	0.239	0.181	0.149	0.277	0.179	0.136	0.420	0.262	0.193
Mean over p	**0.158**	**0.118**	**0.098**	**0.224**	**0.164**	**0.133**	**0.272**	**0.170**	**0.126**	**0.420**	**0.256**	**0.186**
Median over p	0.159	0.119	0.099	0.225	0.166	0.134	0.273	0.172	0.128	0.421	0.258	0.188
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.361	0.297	0.253	0.481	0.382	0.319	0.507	0.362	0.282	0.732	0.490	0.370
$p = 0.10$	0.360	0.297	0.253	0.482	0.384	0.320	**0.487**	0.348	0.270	0.728	0.490	0.368
$p = 0.15$	0.367	0.304	0.259	0.488	0.390	0.326	0.494	0.354	0.276	0.718	0.488	0.370
$p = 0.20$	0.369	0.305	0.260	0.491	0.392	0.328	0.497	0.356	0.278	0.717	0.490	0.372
$p = 0.25$	0.372	0.308	0.263	0.493	0.395	0.331	0.499	0.359	0.280	0.720	0.495	0.376
Mean over p	**0.358**	**0.296**	**0.252**	**0.477**	**0.380**	**0.318**	**0.487**	**0.347**	**0.270**	0.705	0.475	0.359
Median over p	0.361	0.298	0.254	0.478	0.381	**0.318**	0.488	0.348	0.271	**0.702**	**0.474**	**0.358**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.230	0.177	0.149	0.327	0.246	0.201	0.389	0.256	0.197	0.612	0.386	0.286
$p = 0.10$	0.215	0.166	0.139	0.309	0.231	0.188	0.362	0.236	0.180	0.566	0.356	0.263
$p = 0.15$	0.217	0.169	0.142	0.312	0.237	0.194	0.356	0.233	0.178	0.570	0.368	0.276
$p = 0.20$	0.226	0.177	0.149	0.319	0.243	0.200	0.367	0.243	0.187	0.565	0.366	0.275
$p = 0.25$	0.232	0.185	0.155	0.325	0.251	0.209	0.359	0.239	0.184	0.565	0.370	0.279
Mean over p	**0.206**	**0.158**	**0.132**	**0.298**	**0.223**	**0.182**	**0.348**	**0.224**	**0.170**	**0.548**	**0.344**	**0.254**
Median over p	0.208	0.160	0.134	0.299	0.224	0.183	0.350	0.226	0.172	0.552	0.351	0.261

Table A9. VaR forecasting results with the QbSD approach when $\theta = 0$ and $T = 2500$

	$v = 20,\ \lambda = 0$			$v = 20,\ \lambda = -0.5$			$v = 5,\ \lambda = 0$			$v = 5,\ \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.078	0.059	0.046	0.100	0.073	0.054	0.116	0.075	0.054	0.150	0.095	0.063
$p = 0.10$	0.078	0.058	0.045	0.098	0.072	**0.052**	0.112	0.072	0.050	0.147	0.090	0.059
$p = 0.15$	0.080	0.059	0.045	0.100	0.073	0.053	0.115	0.071	0.050	0.149	0.092	0.060
$p = 0.20$	0.082	0.061	0.048	0.104	0.074	0.056	0.115	0.073	0.052	0.151	0.092	0.061
$p = 0.25$	0.086	0.067	0.052	0.107	0.079	0.060	0.120	0.078	0.056	0.154	0.095	0.063
Mean over p	**0.076**	**0.056**	**0.043**	**0.097**	**0.070**	**0.052**	**0.111**	**0.070**	**0.049**	**0.145**	**0.089**	**0.058**
Median over p	0.077	0.057	0.044	0.098	0.071	**0.052**	0.112	0.071	0.050	0.146	0.090	0.059
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.086	0.066	0.052	0.104	0.078	0.058	0.122	0.081	0.059	0.155	0.101	0.067
$p = 0.10$	0.085	0.065	0.051	0.103	0.077	0.057	0.117	0.078	0.055	0.149	0.094	0.062
$p = 0.15$	0.087	0.067	0.053	0.105	0.078	0.058	0.119	0.078	0.057	0.152	0.096	0.064
$p = 0.20$	0.092	0.070	0.056	0.110	0.081	0.062	0.124	0.082	0.060	0.155	0.097	0.065
$p = 0.25$	0.097	0.076	0.059	0.115	0.089	0.067	0.129	0.086	0.062	0.158	0.102	0.069
Mean over p	**0.082**	**0.063**	**0.049**	**0.101**	**0.076**	**0.056**	**0.115**	**0.075**	**0.054**	**0.146**	**0.093**	**0.061**
Median over p	0.083	**0.063**	0.050	**0.101**	**0.076**	**0.056**	0.116	0.076	**0.054**	0.148	0.094	**0.061**
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.103	0.077	0.060	0.130	0.094	0.070	0.161	0.105	0.075	0.201	0.126	0.085
$p = 0.10$	0.102	0.075	0.058	0.127	0.090	0.067	0.154	0.100	0.071	**0.194**	0.118	**0.078**
$p = 0.15$	0.105	0.078	0.060	0.130	0.094	0.070	0.158	0.101	0.072	0.200	0.122	0.081
$p = 0.20$	0.108	0.081	0.063	0.135	0.096	0.072	0.161	0.103	0.075	0.201	0.121	0.081
$p = 0.25$	0.114	0.087	0.069	0.138	0.103	0.078	0.163	0.107	0.077	0.206	0.128	0.086
Mean over p	**0.099**	**0.074**	**0.057**	**0.126**	**0.090**	**0.067**	**0.152**	**0.097**	**0.069**	**0.194**	**0.118**	**0.078**
Median over p	0.101	**0.074**	**0.057**	0.127	0.091	0.068	0.154	0.099	0.070	0.195	0.119	**0.078**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.113	0.086	0.068	0.137	0.101	0.076	0.167	0.111	0.082	0.211	0.135	0.091
$p = 0.10$	0.112	0.085	0.068	0.134	0.098	0.074	0.164	0.108	0.079	0.201	0.124	0.083
$p = 0.15$	0.116	0.089	0.072	0.138	0.102	0.077	0.165	0.109	0.081	0.204	0.127	0.086
$p = 0.20$	0.124	0.096	0.076	0.145	0.106	0.081	0.176	0.119	0.087	0.207	0.127	0.086
$p = 0.25$	0.129	0.100	0.080	0.152	0.115	0.088	0.178	0.119	0.087	0.213	0.135	0.091
Mean over p	**0.108**	**0.083**	**0.065**	**0.131**	**0.096**	0.073	**0.160**	**0.105**	**0.076**	**0.197**	**0.122**	**0.082**
Median over p	0.110	0.084	0.067	0.132	**0.096**	**0.072**	0.162	0.106	0.077	0.199	0.123	**0.082**

Table A10. ES forecasting results with the QbSD approach when $\theta = 0$ and $T = 2500$

	$v = 20,\ \lambda = 0$			$v = 20,\ \lambda = -0.5$			$v = 5,\ \lambda = 0$			$v = 5,\ \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.106	0.080	0.066	0.140	0.104	0.084	0.203	0.131	0.097	0.277	0.173	0.128
$p = 0.10$	0.104	0.078	0.064	0.137	**0.101**	**0.081**	0.197	0.126	0.092	**0.271**	0.167	0.122
$p = 0.15$	0.104	0.079	0.065	0.138	0.102	0.082	0.196	0.125	0.092	0.272	0.169	0.124
$p = 0.20$	0.106	0.081	0.066	0.140	0.105	0.084	0.198	0.127	0.093	0.273	0.171	0.125
$p = 0.25$	0.112	0.087	0.071	0.145	0.109	0.088	0.204	0.131	0.096	0.277	0.173	0.127
Mean over p	**0.102**	**0.077**	**0.062**	**0.136**	**0.101**	**0.081**	**0.195**	**0.124**	**0.091**	**0.271**	**0.168**	**0.123**
Median over p	**0.102**	0.078	0.063	0.137	**0.101**	0.082	0.196	0.125	0.092	**0.271**	0.169	0.124
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.113	0.088	0.072	0.143	0.108	0.087	0.207	0.138	0.103	0.284	0.180	0.132
$p = 0.10$	0.111	0.086	0.071	0.140	0.105	0.084	0.202	0.133	0.098	0.272	0.170	0.123
$p = 0.15$	0.112	0.088	0.072	0.142	0.107	0.086	0.200	0.132	0.098	0.272	0.170	0.125
$p = 0.20$	0.115	0.091	0.075	0.145	0.111	0.090	0.201	0.135	0.101	0.274	0.172	0.126
$p = 0.25$	0.122	0.097	0.080	0.152	0.117	0.095	0.208	0.140	0.104	0.279	0.177	0.129
Mean over p	**0.107**	**0.083**	**0.068**	**0.137**	**0.103**	**0.083**	**0.197**	**0.130**	**0.096**	**0.271**	**0.168**	**0.123**
Median over p	0.108	0.084	0.069	0.138	0.104	0.084	0.198	0.131	0.097	0.272	0.169	0.124
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.135	0.105	0.086	0.177	0.132	0.106	0.263	0.177	0.132	0.349	0.224	0.165
$p = 0.10$	0.132	0.102	0.083	0.173	**0.128**	**0.102**	0.254	0.168	0.125	**0.340**	**0.215**	0.157
$p = 0.15$	0.135	0.104	0.086	0.176	0.131	0.106	0.257	0.170	0.127	0.344	0.220	0.162
$p = 0.20$	0.138	0.108	0.088	0.181	0.135	0.109	0.261	0.174	0.131	0.344	0.219	0.161
$p = 0.25$	0.144	0.114	0.094	0.186	0.140	0.114	0.265	0.176	0.132	0.351	0.225	0.166
Mean over p	**0.130**	**0.100**	**0.081**	**0.173**	**0.128**	0.103	**0.254**	**0.167**	**0.124**	**0.340**	**0.215**	**0.158**
Median over p	0.131	0.101	0.082	0.174	0.129	0.104	0.255	0.169	0.126	**0.340**	0.216	0.159
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.146	0.114	0.095	0.182	0.138	0.111	0.269	0.184	0.139	0.359	0.232	0.172
$p = 0.10$	0.144	0.112	0.093	0.179	0.135	0.108	0.265	0.180	0.135	0.346	0.220	0.161
$p = 0.15$	0.147	0.116	0.097	0.182	0.139	0.112	0.265	0.180	0.136	0.346	0.222	0.164
$p = 0.20$	0.154	0.123	0.103	0.188	0.145	0.116	0.275	0.191	0.145	0.349	0.224	0.164
$p = 0.25$	0.159	0.128	0.107	0.196	0.153	0.124	0.277	0.193	0.146	0.355	0.230	0.169
Mean over p	**0.139**	**0.110**	**0.091**	**0.176**	**0.134**	**0.107**	**0.261**	**0.177**	**0.133**	**0.342**	**0.218**	**0.159**
Median over p	0.142	0.112	0.093	0.177	**0.134**	**0.107**	0.262	0.178	0.134	0.343	0.219	0.160

Table A11. VaR forecasting results with the QbSD approach when $\theta = 0.5$ and $T = 2500$

	$v=20, \lambda=0$			$v=20, \lambda=-0.5$			$v=5, \lambda=0$			$v=5, \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	**0.212**	**0.174**	**0.141**	**0.262**	0.206	**0.162**	**0.229**	0.171	0.130	**0.293**	0.203	**0.147**
$p = 0.10$	0.214	**0.174**	**0.141**	**0.262**	**0.205**	**0.162**	0.230	0.169	0.129	**0.293**	**0.202**	**0.147**
$p = 0.15$	0.217	0.176	0.144	0.266	0.208	0.164	0.233	0.172	0.132	0.297	0.206	0.150
$p = 0.20$	0.217	0.178	0.145	0.265	0.209	0.165	0.235	0.173	0.133	0.298	0.205	0.150
$p = 0.25$	0.222	0.180	0.147	0.270	0.212	0.169	0.239	0.176	0.135	0.303	0.209	0.153
Mean over p	0.214	**0.174**	0.142	0.263	0.206	0.163	0.231	**0.170**	**0.130**	**0.293**	**0.202**	**0.147**
Median over p	0.216	0.175	0.143	0.264	0.206	0.164	0.232	0.172	0.131	0.294	0.203	0.148
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.099	**0.075**	0.061	0.138	0.103	0.079	0.135	0.092	0.068	0.200	0.132	0.092
$p = 0.10$	0.099	0.076	0.059	0.136	**0.102**	**0.077**	0.132	0.090	0.066	0.191	0.125	0.086
$p = 0.15$	0.101	0.080	0.063	0.138	0.105	0.081	0.133	0.090	0.066	0.193	0.128	0.088
$p = 0.20$	0.106	0.083	0.067	0.144	0.110	0.084	0.139	0.095	0.070	0.201	0.133	0.091
$p = 0.25$	0.114	0.090	0.072	0.153	0.118	0.091	0.145	0.099	0.072	0.206	0.135	0.095
Mean over p	**0.096**	**0.075**	**0.059**	**0.133**	**0.102**	0.078	**0.129**	**0.087**	**0.063**	**0.188**	**0.124**	**0.085**
Median over p	0.097	0.076	0.060	0.136	0.103	0.079	0.130	0.088	0.064	0.190	0.126	0.087
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	**0.288**	0.234	0.191	0.365	0.285	0.223	0.335	0.244	0.186	0.436	0.300	0.216
$p = 0.10$	0.287	**0.232**	**0.189**	**0.362**	**0.282**	**0.222**	0.326	0.238	**0.182**	0.431	0.296	0.213
$p = 0.15$	0.292	0.237	0.194	0.367	0.287	0.226	0.330	0.242	0.186	0.436	0.299	0.216
$p = 0.20$	0.294	0.238	0.195	0.366	0.287	0.226	0.332	0.242	0.186	0.437	0.298	0.216
$p = 0.25$	0.303	0.245	0.201	0.375	0.292	0.231	0.343	0.252	0.193	0.449	0.306	0.220
Mean over p	0.289	0.234	0.192	0.363	0.283	0.223	**0.328**	**0.240**	0.184	**0.431**	**0.295**	**0.212**
Median over p	0.290	0.235	0.192	0.365	0.284	0.224	0.329	0.241	0.184	0.432	0.296	0.213
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.134	**0.101**	0.082	0.191	**0.140**	0.109	0.194	0.129	0.096	0.300	0.199	0.140
$p = 0.10$	0.133	**0.101**	**0.080**	**0.188**	**0.140**	**0.107**	0.188	0.127	0.092	0.284	**0.184**	0.126
$p = 0.15$	0.139	0.107	0.086	0.196	0.147	0.113	0.189	0.126	0.093	0.301	0.195	0.132
$p = 0.20$	0.146	0.114	0.091	0.206	0.153	0.117	0.201	0.135	0.099	0.312	0.199	0.136
$p = 0.25$	0.161	0.127	0.102	0.217	0.166	0.128	0.215	0.146	0.108	0.321	0.207	0.146
Mean over p	**0.131**	**0.101**	**0.080**	**0.188**	**0.140**	**0.107**	**0.186**	**0.123**	**0.090**	**0.288**	0.185	**0.127**
Median over p	0.133	0.102	0.082	0.191	0.142	0.109	0.187	0.124	0.091	0.298	0.192	0.132

Table A12. ES forecasting results with the QbSD approach when $\theta = 0.5$ and $T = 2500$

	$v = 20,\ \lambda = 0$			$v = 20,\ \lambda = -0.5$			$v = 5,\ \lambda = 0$			$v = 5,\ \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	**0.250**	**0.212**	0.183	**0.320**	0.263	0.223	0.326	0.242	0.193	0.449	0.317	0.246
$p = 0.10$	0.251	**0.212**	**0.182**	0.319	**0.262**	**0.222**	**0.323**	**0.241**	**0.191**	**0.444**	**0.314**	0.243
$p = 0.15$	0.254	0.215	0.185	0.322	0.265	0.224	0.327	0.244	0.194	0.446	0.317	0.245
$p = 0.20$	0.256	0.216	0.186	0.324	0.266	0.224	0.326	0.244	0.194	0.447	0.317	0.245
$p = 0.25$	0.260	0.219	0.188	0.330	0.271	0.229	0.331	0.247	0.197	0.456	0.323	0.249
Mean over p	0.252	**0.212**	0.183	0.321	0.263	0.223	**0.323**	0.242	0.192	0.445	0.315	**0.243**
Median over p	0.252	0.213	0.184	0.322	0.264	0.224	0.324	0.243	0.193	0.446	0.316	0.244
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.128	0.099	0.082	0.182	0.140	0.113	0.226	0.151	0.112	0.348	0.226	0.166
$p = 0.10$	0.125	0.097	0.081	0.179	0.137	0.113	0.216	0.145	0.107	0.328	**0.212**	**0.157**
$p = 0.15$	0.125	0.099	0.083	0.179	0.139	0.115	**0.213**	**0.143**	0.107	0.328	0.213	0.159
$p = 0.20$	0.133	0.105	0.088	0.185	0.143	0.118	0.221	0.150	0.113	0.334	0.218	0.162
$p = 0.25$	0.142	0.113	0.095	0.196	0.154	0.126	0.229	0.156	0.116	0.340	0.224	0.167
Mean over p	**0.123**	**0.095**	**0.080**	**0.177**	**0.135**	**0.112**	0.214	**0.143**	**0.105**	**0.327**	**0.212**	**0.157**
Median over p	0.124	0.096	0.081	**0.177**	0.137	0.113	**0.213**	**0.143**	0.106	0.330	0.214	0.159
Panel B: Root mean squared error												
QbSD-gSAV model												
Individual values of p												
$p = 0.05$	0.338	0.287	0.248	0.441	0.365	0.308	0.457	0.346	0.276	0.640	0.462	0.360
$p = 0.10$	**0.336**	**0.285**	**0.246**	**0.438**	**0.362**	**0.304**	**0.450**	**0.340**	**0.271**	0.630	**0.454**	0.354
$p = 0.15$	0.342	0.290	0.251	0.443	0.367	0.309	0.456	0.345	0.276	0.632	0.458	0.357
$p = 0.20$	0.344	0.292	0.253	0.443	0.367	0.309	0.456	0.345	0.276	0.631	0.458	0.357
$p = 0.25$	0.354	0.301	0.260	0.453	0.376	0.316	0.470	0.356	0.286	0.645	0.468	0.365
Mean over p	0.339	0.288	0.249	0.439	0.364	0.306	0.453	0.342	0.274	**0.627**	**0.454**	**0.353**
Median over p	0.339	0.288	0.249	0.441	0.365	0.307	0.455	0.344	0.275	0.629	0.455	**0.355**
QbSD-gAS model												
Individual values of p												
$p = 0.05$	0.170	0.133	0.111	0.246	0.190	0.156	0.304	0.209	0.156	0.485	0.326	0.244
$p = 0.10$	0.167	**0.130**	**0.109**	0.242	0.188	0.155	0.293	0.201	0.150	**0.450**	**0.301**	**0.225**
$p = 0.15$	0.172	0.136	0.115	0.246	0.193	0.160	**0.288**	**0.199**	0.150	0.457	0.312	0.235
$p = 0.20$	0.181	0.145	0.122	0.257	0.202	0.166	0.301	0.211	0.160	0.471	0.322	0.243
$p = 0.25$	0.195	0.159	0.136	0.269	0.215	0.178	0.318	0.225	0.171	0.481	0.332	0.253
Mean over p	**0.165**	**0.130**	0.110	**0.240**	**0.187**	**0.154**	0.290	**0.199**	**0.149**	0.452	0.304	0.228
Median over p	0.167	0.131	0.111	0.242	0.189	0.156	0.290	**0.199**	**0.149**	0.461	0.313	0.236

Section B

Tables B1–B8 report one-step-ahead VaR/ES forecast errors (MAE, RMSE) for the proposed QbSD models (gSAV, gAS) and benchmarks (skew-t GARCH/GJR/EGARCH, AL-based joint VaR-ES, and GAS) across data-generating designs varying by $T \in \{250, 2500\}$, $v \in \{5, 20\}$, $\lambda \in \{0, -0.5\}$, $\theta \in \{0, 0.5\}$, and $\alpha \in \{0.01, 0.025, 0.05\}$. Minima are bolded.

For $T = 250$ without leverage (Tables B1–B2), conventional GARCH-type models dominate. With symmetric innovations, Normal/Student-t GARCH/GJR achieve most minima; under left-skew, their skew-t counterparts lead. QbSD models are competitive but rarely best, while AL-based estimators and GAS trail.

Introducing leverage at $T = 250$ (Tables B3–B4) shifts the lead within the GARCH family: EGARCH frequently attains the lowest errors in symmetric cases, and skew-t GARCH/GJR often prevail under skewed or heavy-tailed DGPs. QbSD-gAS improves relative to QbSD-gSAV but seldom displaces the top GARCH variants at this sample size.

With larger samples and no leverage (Tables B5–B6, $T = 2500$, $\theta = 0$), GARCH/GJR remain the primary winners. Student-t specifications account for many minima under symmetry and heavy tails, whereas skew-t versions excel when $\lambda = -0.5$. QbSD narrows the gap but generally does not surpass the best GARCH-type models here.

With larger samples under leverage (Tables B7–B8, $T = 2500$, $\theta = 0.5$), QbSD-gAS delivers many of the lowest ES MAE/RMSE values across α and (v, λ) settings, outperforming AL and GAS and often surpassing standard GARCH models. Two caveats remain: (i) under symmetric innovations, EGARCH can match or exceed QbSD-gAS for ES RMSE at some α (e.g., $v = 20$, $\lambda = 0$ at 2.5% and 5%); and (ii) for VaR the best model can shift with tail level—EGARCH under symmetry and skew-t GJR-GARCH/GARCH under skewed, heavy-tailed designs—while QbSD-gAS remains the most robust ES performer under leverage.

Overall, while GARCH-type models (especially skew-t and EGARCH) are very strong at $T = 250$ and in several $T = 2500$ designs without leverage, QbSD-gAS becomes particularly effective for ES once leverage is present and the sample is large, delivering broad gains except in the most extreme skew/heavy-tail scenario. AL-based estimators and GAS underperform across configurations.

Table B1. VaR forecasting results when $\theta = 0$ and $T = 250$

	$v = 20,\ \lambda = 0$			$v = 20,\ \lambda = -0.5$			$v = 5,\ \lambda = 0$			$v = 5,\ \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.218	0.165	0.131	0.281	0.198	0.154	0.300	0.193	0.138	0.403	0.239	0.167
QbSD-gAS	0.241	0.182	0.147	0.304	0.216	0.169	0.318	0.208	0.153	0.431	0.259	0.180
AL approach												
$AL_{Mult.}$-SAV	0.314	0.225	0.174	0.435	0.307	0.227	0.473	0.287	0.197	0.661	0.414	0.267
AL_{AR}-SAV	0.313	0.228	0.178	0.437	0.308	0.235	0.462	0.287	0.201	0.645	0.412	0.266
$AL_{Mult.}$-AS	0.378	0.267	0.200	0.503	0.355	0.260	0.537	0.338	0.233	0.770	0.479	0.318
AL_{AR}-AS	0.378	0.271	0.202	0.499	0.362	0.268	0.532	0.339	0.232	0.747	0.471	0.316
GAS	0.351	0.264	0.202	0.448	0.315	0.232	0.717	0.333	0.231	0.595	0.377	0.284
GARCH												
Normal	**0.146**	**0.114**	0.094	0.479	0.311	0.196	0.284	0.158	0.133	0.776	0.390	0.199
Student-t	0.151	0.115	**0.093**	0.415	0.298	0.205	0.227	**0.145**	**0.104**	0.618	0.378	0.225
Skew-t	0.165	0.124	0.097	**0.201**	**0.138**	**0.101**	0.245	0.153	0.107	**0.329**	**0.181**	**0.111**
GJR-GARCH												
Normal	0.147	0.115	0.095	0.479	0.311	0.196	0.286	0.160	0.136	0.776	0.390	0.197
Student-t	0.149	0.114	**0.092**	0.414	0.297	0.206	**0.227**	**0.145**	**0.104**	0.618	0.377	0.225
Skew-t	0.165	0.124	0.098	0.203	0.140	0.102	0.247	0.156	0.110	0.338	0.188	0.116
EGARCH	0.188	0.149	0.123	0.515	0.342	0.224	0.343	0.209	0.167	0.835	0.447	0.246
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.281	0.216	0.173	0.361	0.259	0.206	0.400	0.265	0.196	0.568	0.343	0.245
QbSD-gAS	0.308	0.235	0.192	0.392	0.284	0.225	0.435	0.283	0.215	0.655	0.384	0.281
AL approach												
$AL_{Mult.}$-SAV	0.416	0.301	0.237	0.600	0.419	0.318	0.694	0.407	0.292	1.039	0.684	0.433
AL_{AR}-SAV	0.407	0.306	0.245	0.597	0.424	0.332	0.649	0.407	0.296	0.952	0.616	0.413
$AL_{Mult.}$-AS	0.492	0.363	0.267	0.661	0.508	0.349	0.733	0.462	0.330	1.157	0.720	0.725
AL_{AR}-AS	0.493	0.368	0.272	0.650	0.521	0.354	0.727	0.485	0.350	1.031	0.690	0.472
GAS	0.714	0.410	0.282	0.815	0.597	0.461	8.371	0.750	0.488	1.010	0.708	0.773
GARCH												
Normal	**0.188**	**0.152**	0.127	0.515	0.345	0.227	0.341	0.225	0.203	0.826	0.444	0.261
Student-t	0.203	0.156	0.126	0.460	0.331	0.233	0.321	0.208	0.150	0.676	0.414	0.257
Skew-t	0.221	0.167	0.131	**0.295**	**0.206**	**0.147**	0.346	0.218	0.155	**0.485**	**0.278**	**0.177**
GJR-GARCH												
Normal	0.191	0.154	0.129	0.515	0.345	0.227	0.346	0.232	0.211	0.823	0.441	0.258
Student-t	0.201	0.155	**0.125**	0.460	0.331	0.235	**0.320**	**0.207**	**0.150**	0.676	0.414	0.257
Skew-t	0.221	0.167	0.132	0.285	0.199	0.145	0.345	0.219	0.157	0.497	0.288	0.184
EGARCH	0.253	0.204	0.168	0.570	0.393	0.270	0.432	0.296	0.249	0.909	0.523	0.330

Table B2. ES forecasting results when $\theta = 0$ and $T = 250$

	$v = 20$, $\lambda = 0$			$v = 20$, $\lambda = -0.5$			$v = 5$, $\lambda = 0$			$v = 5$, $\lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.300	0.219	0.174	0.407	0.283	0.219	0.522	0.322	0.229	0.745	0.437	0.298
QbSD-gAS	0.322	0.241	0.195	0.432	0.306	0.238	0.545	0.342	0.246	0.779	0.466	0.321
AL approach												
$AL_{Mult.}$-SAV	0.391	0.280	0.211	0.551	0.389	0.294	0.687	0.417	0.277	0.995	0.626	0.410
AL_{AR}-SAV	0.384	0.291	0.227	0.536	0.395	0.303	0.644	0.428	0.297	0.925	0.622	0.427
$AL_{Mult.}$-AS	0.467	0.333	0.248	0.636	0.452	0.334	0.765	0.472	0.328	1.111	0.691	0.472
AL_{AR}-AS	0.458	0.324	0.252	0.615	0.438	0.325	0.729	0.460	0.323	1.027	0.640	0.440
GAS	0.440	0.330	0.267	0.577	0.416	0.323	0.980	0.483	0.337	0.938	0.616	0.458
GARCH												
Normal	**0.197**	**0.152**	**0.124**	0.687	0.502	0.372	0.636	0.351	0.215	1.461	0.904	0.577
Student-t	0.206	0.158	0.128	0.535	0.422	0.334	**0.410**	**0.265**	**0.188**	0.984	0.678	0.483
Skew-t	0.225	0.171	0.138	**0.302**	**0.215**	**0.163**	0.446	0.286	0.201	**0.654**	**0.401**	**0.265**
GJR-GARCH												
Normal	0.199	0.153	0.125	0.686	0.502	0.372	0.638	0.353	0.217	1.462	0.904	0.577
Student-t	0.204	0.156	0.127	0.535	0.421	0.334	**0.410**	**0.265**	**0.188**	0.982	0.676	0.482
Skew-t	0.225	0.172	0.139	0.304	0.217	0.163	0.446	0.288	0.203	0.642	0.393	0.260
EGARCH	0.243	0.193	0.162	0.727	0.538	0.404	0.699	0.409	0.268	1.524	0.962	0.632
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.378	0.282	0.227	0.510	0.361	0.281	0.672	0.424	0.307	0.993	0.602	0.417
QbSD-gAS	0.410	0.309	0.253	0.552	0.396	0.311	0.736	0.467	0.340	1.171	0.707	0.493
AL approach												
$AL_{Mult.}$-SAV	0.498	0.363	0.289	0.730	0.519	0.406	0.968	0.564	0.406	1.687	0.985	0.672
AL_{AR}-SAV	0.502	0.384	0.311	0.716	0.528	0.414	0.842	0.589	0.431	1.268	0.884	0.631
$AL_{Mult.}$-AS	0.594	0.437	0.333	0.803	0.604	0.446	0.993	0.624	0.453	1.589	0.994	0.989
AL_{AR}-AS	0.590	0.428	0.340	0.780	0.590	0.435	0.956	0.643	0.462	1.373	0.888	0.708
GAS	0.875	0.494	0.371	0.950	0.740	0.474	9.766	0.979	0.652	1.342	1.124	1.098
GARCH												
Normal	**0.242**	**0.193**	**0.163**	0.724	0.538	0.406	0.689	0.403	0.270	1.515	0.952	0.625
Student-t	0.276	0.212	0.173	0.602	0.471	0.373	0.582	0.376	0.269	1.089	0.743	0.529
Skew-t	0.300	0.230	0.186	0.436	0.316	0.242	0.629	0.406	0.287	**0.959**	**0.591**	**0.397**
GJR-GARCH												
Normal	0.246	0.196	0.165	0.724	0.538	0.406	0.690	0.407	0.275	1.512	0.950	0.622
Student-t	0.274	0.210	0.171	0.602	0.470	0.371	**0.580**	**0.375**	**0.268**	1.093	0.747	0.533
Skew-t	0.299	0.230	0.186	**0.421**	**0.305**	**0.232**	0.622	0.403	0.287	0.974	0.603	0.407
EGARCH	0.318	0.260	0.219	0.784	0.593	0.456	0.779	0.492	0.351	1.599	1.033	0.702

Table B3. VaR forecasting results when $\theta = 0.5$ and $T = 250$

	$v=20, \lambda=0$			$v=20, \lambda=-0.5$			$v=5, \lambda=0$			$v=5, \lambda=-0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.302	0.226	0.185	0.396	0.289	0.229	0.373	0.243	0.181	0.545	0.344	0.240
QbSD-gAS	0.262	0.196	0.159	0.357	0.260	0.204	0.356	0.232	0.172	0.547	0.331	0.235
AL approach												
$AL_{Mult.}$-SAV	0.420	0.296	0.234	0.592	0.419	0.308	0.573	0.354	0.244	0.866	0.554	0.357
AL_{AR}-SAV	0.419	0.301	0.239	0.594	0.421	0.318	0.567	0.350	0.251	0.823	0.528	0.348
$AL_{Mult.}$-AS	0.441	0.305	0.225	0.656	0.433	0.341	0.612	0.390	0.252	0.954	0.584	0.395
AL_{AR}-AS	0.445	0.318	0.231	0.653	0.448	0.344	0.604	0.387	0.260	0.958	0.603	0.402
GAS	0.483	0.336	0.248	0.756	0.522	0.355	0.591	0.393	0.251	0.946	0.576	0.401
GARCH												
Normal	0.244	0.199	0.165	0.565	0.374	0.244	0.355	0.232	0.195	0.920	0.466	0.241
Student-t	0.255	0.201	0.163	0.488	0.358	0.255	0.320	0.215	**0.159**	0.717	0.447	0.273
Skew-t	0.263	0.205	0.165	**0.294**	**0.214**	**0.165**	0.329	0.220	0.161	**0.384**	0.233	0.157
GJR-GARCH												
Normal	0.234	0.190	0.157	0.560	0.369	0.240	0.352	0.223	0.186	0.916	0.464	0.239
Student-t	0.253	0.200	0.162	0.487	0.357	0.254	**0.316**	**0.214**	**0.159**	0.717	0.446	0.272
Skew-t	0.263	0.205	0.165	**0.294**	0.215	**0.165**	0.330	0.223	0.164	**0.384**	**0.232**	**0.156**
EGARCH	**0.213**	**0.172**	**0.143**	0.582	0.386	0.252	0.368	0.226	0.185	0.944	0.507	0.285
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.393	0.304	0.250	0.534	0.403	0.324	0.520	0.344	0.265	1.032	0.690	0.411
QbSD-gAS	0.395	0.290	0.238	0.575	0.417	0.331	0.643	0.411	0.308	1.670	0.782	0.515
AL approach												
$AL_{Mult.}$-SAV	0.561	0.403	0.320	0.820	0.586	0.447	0.867	0.529	0.363	1.663	1.173	0.648
AL_{AR}-SAV	0.555	0.414	0.325	0.832	0.580	0.456	0.838	0.529	0.381	1.290	0.909	0.547
$AL_{Mult.}$-AS	0.594	0.469	0.311	0.908	0.607	0.531	0.925	0.873	0.382	1.332	0.908	0.695
AL_{AR}-AS	0.607	0.493	0.317	0.896	0.634	0.484	0.858	0.816	0.395	1.470	1.022	0.734
GAS	0.758	0.617	0.336	1.258	1.040	0.518	1.344	1.110	0.443	1.883	1.041	0.851
GARCH												
Normal	0.332	0.273	0.229	0.662	0.450	0.305	0.468	0.336	0.292	1.115	0.583	0.319
Student-t	0.348	0.275	0.224	0.586	0.435	0.317	0.454	0.307	**0.228**	0.900	0.588	0.383
Skew-t	0.360	0.283	0.224	**0.412**	**0.300**	**0.228**	0.464	0.312	0.230	0.569	0.352	0.242
GJR-GARCH												
Normal	0.320	0.262	0.219	0.652	0.442	0.297	0.462	0.331	0.289	1.107	0.578	0.317
Student-t	0.347	0.275	0.222	0.433	0.433	0.315	**0.447**	**0.304**	**0.227**	0.899	0.587	0.383
Skew-t	0.360	0.283	0.226	0.413	0.301	0.229	0.474	0.323	0.239	**0.568**	**0.347**	**0.236**
EGARCH	**0.298**	**0.242**	**0.201**	0.681	0.467	0.320	0.481	0.342	0.293	1.106	0.626	0.413

Table B4. ES forecasting results when $\theta = 0.5$ and $T = 250$

	$v = 20$, $\lambda = 0$			$v = 20$, $\lambda = -0.5$			$v = 5$, $\lambda = 0$			$v = 5$, $\lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.396	0.300	0.243	0.540	0.397	0.314	0.633	0.402	0.289	0.984	0.594	0.413
QbSD-gAS	0.348	0.258	0.209	0.497	0.356	0.280	0.595	0.376	0.273	0.979	0.589	0.410
AL approach												
$AL_{Mult.}$-SAV	0.512	0.373	0.294	0.741	0.524	0.398	0.816	0.507	0.352	1.324	0.800	0.557
AL_{AR}-SAV	0.501	0.382	0.307	0.730	0.534	0.426	0.771	0.505	0.362	1.198	0.769	0.537
$AL_{Mult.}$-AS	0.542	0.384	0.285	0.826	0.569	0.439	0.859	0.552	0.362	1.358	0.872	0.599
AL_{AR}-AS	0.530	0.384	0.290	0.806	0.561	0.426	0.818	0.520	0.362	1.287	0.814	0.564
GAS	0.595	0.439	0.328	0.950	0.692	0.484	0.849	0.588	0.382	1.463	0.909	0.633
GARCH												
Normal	0.301	0.249	0.213	0.804	0.591	0.442	0.710	0.417	0.283	1.729	1.071	0.685
Student-t	0.332	0.265	0.219	0.621	0.495	0.396	0.541	0.364	0.269	1.119	0.780	0.561
Skew-t	0.341	0.271	0.221	**0.413**	**0.309**	**0.244**	0.553	0.374	0.275	**0.312**	**0.452**	**0.312**
GJR-GARCH												
Normal	0.292	0.239	0.204	0.799	0.586	0.438	0.713	0.416	0.278	1.724	1.067	0.682
Student-t	0.329	0.262	0.216	0.624	0.494	0.395	**0.531**	**0.359**	**0.266**	1.117	0.778	0.561
Skew-t	0.340	0.271	0.223	0.412	0.310	0.244	0.546	0.372	0.276	0.313	0.454	0.313
EGARCH	**0.271**	**0.218**	**0.185**	0.824	0.609	0.456	0.746	0.436	0.288	1.739	1.090	0.715
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.518	0.395	0.326	0.738	0.540	0.433	0.909	0.566	0.411	2.831	1.513	0.963
QbSD-gAS	0.506	0.389	0.315	0.792	0.577	0.458	1.101	0.700	0.503	3.575	1.994	1.261
AL approach												
$AL_{Mult.}$-SAV	0.673	0.498	0.399	1.004	0.726	0.578	1.297	0.826	0.555	3.951	1.726	1.266
AL_{AR}-SAV	0.653	0.506	0.405	0.986	0.703	0.572	1.072	0.711	0.510	1.737	1.143	0.771
$AL_{Mult.}$-AS	0.717	0.542	0.391	1.109	0.775	0.659	1.269	1.080	0.562	2.212	1.712	1.136
AL_{AR}-AS	0.707	0.550	0.382	1.084	0.757	0.567	1.120	0.875	0.515	1.984	1.306	0.803
GAS	0.905	0.725	0.446	1.480	1.229	0.711	1.688	1.362	0.626	3.622	1.643	1.367
GARCH												
Normal	0.403	0.337	0.291	0.921	0.689	0.525	0.835	0.529	0.385	2.058	1.289	0.838
Student-t	0.451	0.359	0.299	0.743	0.594	0.480	0.787	0.522	0.383	1.358	0.966	0.714
Skew-t	0.466	0.371	0.303	**0.583**	**0.435**	**0.342**	0.796	0.531	0.390	**1.054**	**0.670**	**0.466**
GJR-GARCH												
Normal	0.390	0.325	0.280	0.910	0.680	0.517	0.827	0.522	0.379	2.049	1.282	0.832
Student-t	0.450	0.359	0.298	0.750	0.592	0.478	**0.761**	**0.510**	**0.377**	1.356	0.966	0.713
Skew-t	0.465	0.371	0.306	0.586	0.436	0.343	0.792	0.536	0.398	1.061	0.674	0.465
EGARCH	**0.369**	**0.304**	**0.259**	0.940	0.707	0.542	0.853	0.543	0.395	2.007	1.268	0.845

Table B5. VaR forecasting results when $\theta = 0$ and $T = 2500$

	$v = 20, \lambda = 0$			$v = 20, \lambda = -0.5$			$v = 5, \lambda = 0$			$v = 5, \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.078	0.057	0.046	0.099	0.070	0.054	0.112	0.071	0.052	0.150	0.090	0.062
QbSD-gAS	0.088	0.066	0.053	0.107	0.077	0.059	0.121	0.079	0.058	0.154	0.095	0.065
AL approach												
$AL_{Mult.}$-SAV	0.109	0.071	0.052	0.148	0.093	0.067	0.168	0.092	0.060	0.235	0.127	0.079
AL_{AR}-SAV	0.111	0.073	0.054	0.150	0.097	0.070	0.169	0.097	0.064	0.242	0.133	0.088
$AL_{Mult.}$-AS	0.124	0.081	0.058	0.173	0.108	0.077	0.192	0.107	0.068	0.278	0.149	0.092
AL_{AR}-AS	0.125	0.083	0.060	0.175	0.110	0.082	0.197	0.111	0.071	0.284	0.159	0.103
GAS	0.275	0.191	0.143	0.323	0.203	0.139	0.326	0.210	0.144	0.374	0.221	0.135
GARCH												
Normal	0.074	0.045	0.035	0.466	0.294	0.172	0.225	0.068	0.077	0.747	0.348	0.135
Student-t	**0.055**	**0.042**	**0.034**	0.380	0.276	0.189	**0.078**	**0.053**	**0.040**	0.592	0.361	0.211
Skew-t	0.058	0.045	0.036	**0.076**	**0.053**	**0.040**	0.082	0.055	0.041	0.119	0.072	0.048
GJR-GARCH												
Normal	0.074	0.045	0.035	0.466	0.294	0.172	0.225	0.069	0.077	0.745	0.347	0.135
Student-t	0.055	0.042	0.034	0.380	0.276	0.189	0.078	0.053	0.040	0.592	0.361	0.211
Skew-t	0.059	0.045	0.036	0.076	0.053	0.040	0.082	0.055	0.041	**0.112**	**0.068**	**0.045**
EGARCH	0.075	0.048	0.040	0.472	0.299	0.176	0.230	0.070	0.080	0.761	0.355	0.132
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.104	0.076	0.061	0.131	0.092	0.071	0.153	0.098	0.073	0.202	0.124	0.089
QbSD-gAS	0.119	0.090	0.073	0.143	0.103	0.080	0.167	0.111	0.083	0.221	0.142	0.100
AL approach												
$AL_{Mult.}$-SAV	0.150	0.094	0.069	0.209	0.124	0.089	0.247	0.129	0.084	0.363	0.188	0.117
AL_{AR}-SAV	0.153	0.098	0.071	0.214	0.131	0.094	0.255	0.140	0.096	0.375	0.201	0.146
$AL_{Mult.}$-AS	0.176	0.111	0.078	0.246	0.149	0.106	0.285	0.154	0.096	0.424	0.220	0.134
AL_{AR}-AS	0.177	0.113	0.081	0.252	0.153	0.115	0.303	0.165	0.103	0.451	0.238	0.161
GAS	0.355	0.256	0.191	0.452	0.267	0.182	0.506	0.305	0.206	0.524	0.305	0.185
GARCH												
Normal	0.090	0.061	0.051	0.478	0.304	0.182	0.247	0.101	0.115	0.774	0.370	0.162
Student-t	**0.077**	**0.060**	**0.049**	0.394	0.286	0.197	**0.119**	**0.084**	**0.064**	0.610	0.374	0.221
Skew-t	0.081	0.063	0.051	**0.107**	**0.078**	**0.059**	0.125	0.087	0.065	0.182	0.143	0.096
GJR-GARCH												
Normal	0.091	0.062	0.052	0.478	0.304	0.182	0.248	0.102	0.115	0.769	0.367	0.163
Student-t	0.077	0.060	0.049	0.394	0.286	0.197	0.119	0.084	0.064	0.610	0.374	0.221
Skew-t	0.081	0.063	0.051	0.107	0.078	0.059	0.125	0.087	0.065	**0.184**	**0.118**	**0.081**
EGARCH	0.103	0.069	0.055	0.490	0.314	0.190	0.268	0.105	0.103	0.805	0.389	0.166

Table B6. ES forecasting results when $\theta = 0$ and $T = 2500$

	$v = 20, \lambda = 0$			$v = 20, \lambda = -0.5$			$v = 5, \lambda = 0$			$v = 5, \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.104	0.079	0.064	0.138	0.100	0.081	0.199	0.126	0.092	0.278	0.172	0.123
QbSD-gAS	0.113	0.086	0.072	0.144	0.107	0.087	0.202	0.131	0.097	0.280	0.173	0.126
AL approach												
$AL_{Mult.}$-SAV	0.134	0.088	0.066	0.187	0.118	0.088	0.253	0.140	0.090	0.368	0.199	0.128
AL_{AR}-SAV	0.157	0.109	0.082	0.216	0.148	0.105	0.297	0.175	0.115	0.446	0.253	0.157
$AL_{Mult.}$-AS	0.156	0.100	0.073	0.219	0.136	0.101	0.286	0.158	0.101	0.417	0.229	0.146
AL_{AR}-AS	0.173	0.116	0.087	0.243	0.157	0.115	0.335	0.191	0.122	0.471	0.269	0.166
GAS	0.332	0.240	0.188	0.390	0.261	0.190	0.442	0.295	0.214	0.535	0.323	0.206
GARCH												
Normal	0.136	0.083	0.056	0.676	0.490	0.357	0.618	0.308	0.148	1.443	0.880	0.546
Student-t	**0.075**	**0.057**	**0.047**	0.473	0.382	0.305	0.132	**0.088**	**0.065**	0.932	0.646	0.461
Skew-t	0.079	0.061	0.049	0.110	0.080	0.062	0.137	0.092	0.068	0.220	0.140	0.097
GJR-GARCH												
Normal	0.136	0.084	0.056	0.676	0.490	0.357	0.619	0.309	0.148	1.440	0.877	0.544
Student-t	0.076	0.057	0.047	0.473	0.382	0.305	**0.132**	0.088	0.065	0.932	0.646	0.471
Skew-t	0.079	0.061	0.049	**0.110**	**0.080**	**0.062**	0.137	0.092	0.068	**0.206**	**0.132**	**0.091**
EGARCH	0.138	0.084	0.058	0.683	0.496	0.363	0.630	0.317	0.150	1.460	0.894	0.557
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.138	0.105	0.086	0.180	0.132	0.107	0.257	0.169	0.125	0.355	0.225	0.163
QbSD-gAS	0.152	0.119	0.099	0.189	0.143	0.116	0.266	0.179	0.135	0.369	0.237	0.178
AL approach												
$AL_{Mult.}$-SAV	0.184	0.118	0.087	0.260	0.160	0.119	0.350	0.192	0.125	0.531	0.286	0.187
AL_{AR}-SAV	0.217	0.154	0.110	0.306	0.206	0.142	0.480	0.267	0.162	0.751	0.421	0.223
$AL_{Mult.}$-AS	0.214	0.139	0.099	0.303	0.188	0.142	0.397	0.223	0.143	0.594	0.325	0.215
AL_{AR}-AS	0.245	0.163	0.117	0.348	0.219	0.157	0.531	0.298	0.167	0.764	0.419	0.236
GAS	0.434	0.321	0.250	0.545	0.346	0.282	0.668	0.426	0.300	0.750	0.459	0.282
GARCH												
Normal	0.152	0.099	0.071	0.691	0.502	0.368	0.641	0.328	0.169	1.486	0.909	0.569
Student-t	**0.102**	**0.079**	**0.066**	0.492	0.396	0.316	**0.190**	**0.132**	**0.100**	0.962	0.667	0.476
Skew-t	0.107	0.084	0.069	**0.150**	0.113	**0.089**	0.199	0.139	0.105	0.399	0.263	0.187
GJR-GARCH												
Normal	0.153	0.100	0.072	0.691	0.502	0.368	0.642	0.329	0.170	1.479	0.903	0.565
Student-t	0.102	0.080	0.066	0.492	0.396	0.316	0.190	0.132	0.100	0.962	0.667	0.476
Skew-t	0.107	0.084	0.069	0.150	**0.113**	0.089	0.198	0.138	0.105	**0.321**	**0.212**	**0.152**
EGARCH	0.168	0.113	0.082	0.705	0.514	0.378	0.669	0.351	0.186	1.524	0.940	0.594

Table B7. VaR forecasting results when $\theta = 0.5$ and $T = 2500$

	$v = 20, \lambda = 0$			$v = 20, \lambda = -0.5$			$v = 5, \lambda = 0$			$v = 5, \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.209	0.168	0.137	0.258	0.202	0.161	0.226	0.166	0.127	0.296	0.204	0.150
QbSD-gAS	0.096	0.072	0.057	**0.135**	**0.099**	**0.077**	**0.130**	0.086	**0.063**	**0.196**	**0.126**	**0.088**
AL approach												
$AL_{Mult.}$-SAV	0.227	0.174	0.140	0.292	0.212	0.167	0.267	0.176	0.130	0.373	0.228	0.157
AL_{AR}-SAV	0.228	0.175	0.142	0.292	0.215	0.170	0.267	0.179	0.134	0.393	0.238	0.171
$AL_{Mult.}$-AS	0.135	0.090	0.068	0.204	0.132	0.097	0.211	0.114	0.077	0.352	0.180	0.117
AL_{AR}-AS	0.138	0.093	0.072	0.207	0.138	0.107	0.209	0.120	0.085	0.360	0.197	0.142
GAS	0.370	0.239	0.155	0.597	0.345	0.210	0.430	0.236	0.141	0.720	0.340	0.204
GARCH												
Normal	0.204	0.167	0.139	0.526	0.336	0.207	0.287	0.172	0.154	0.857	0.401	0.168
Student-t	0.206	0.168	0.138	0.427	0.316	0.223	0.226	0.169	0.131	0.672	0.414	0.245
Skew-t	0.207	0.168	0.138	0.201	0.157	0.124	0.227	0.169	0.131	0.221	0.151	0.110
GJR-GARCH												
Normal	0.195	0.158	0.132	0.526	0.336	0.207	0.287	0.164	0.146	0.857	0.401	0.167
Student-t	0.206	0.168	0.138	0.427	0.316	0.223	0.226	0.169	0.131	0.672	0.414	0.245
Skew-t	0.207	0.169	0.138	0.201	0.157	0.124	0.228	0.169	0.131	0.218	0.150	0.109
EGARCH	**0.086**	**0.058**	**0.048**	0.548	0.348	0.206	0.247	**0.079**	0.087	0.893	0.420	0.161
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.283	0.227	0.185	0.357	0.279	0.220	0.317	0.235	0.180	0.447	0.311	0.229
QbSD-gAS	0.133	0.100	0.079	**0.197**	**0.146**	**0.113**	**0.193**	0.134	**0.101**	**0.327**	**0.238**	**0.176**
AL approach												
$AL_{Mult.}$-SAV	0.308	0.234	0.188	0.416	0.296	0.232	0.389	0.252	0.185	0.596	0.349	0.241
AL_{AR}-SAV	0.312	0.237	0.190	0.413	0.304	0.237	0.402	0.263	0.190	0.689	0.381	0.271
$AL_{Mult.}$-AS	0.189	0.128	0.096	0.296	0.193	0.147	0.317	0.178	0.122	0.641	0.299	0.214
AL_{AR}-AS	0.193	0.135	0.105	0.309	0.209	0.174	0.317	0.208	0.146	0.656	0.371	0.345
GAS	0.479	0.319	0.206	0.920	0.461	0.289	0.872	0.324	0.195	1.454	0.521	0.342
GARCH												
Normal	0.273	0.224	0.188	0.616	0.406	0.261	0.388	0.263	0.239	1.018	0.488	0.237
Student-t	0.279	0.226	0.186	0.515	0.386	0.278	0.344	0.256	0.198	0.818	0.511	0.311
Skew-t	0.281	0.227	0.186	0.284	0.219	0.172	0.349	0.258	0.199	0.374	0.250	0.177
GJR-GARCH												
Normal	0.264	0.216	0.182	0.616	0.406	0.261	0.384	0.257	0.233	1.017	0.488	0.236
Student-t	0.279	0.226	0.186	0.515	0.386	0.278	0.343	0.255	0.198	0.819	0.511	0.311
Skew-t	0.281	0.227	0.186	0.284	0.219	0.172	0.350	0.259	0.199	0.364	0.246	0.175
EGARCH	**0.123**	**0.084**	**0.066**	0.637	0.416	0.260	0.311	**0.125**	0.112	1.164	0.593	0.279

Table B8. ES forecasting results when $\theta = 0.5$ and $T = 2500$

	$v = 20, \lambda = 0$			$v = 20, \lambda = -0.5$			$v = 5, \lambda = 0$			$v = 5, \lambda = -0.5$		
$\alpha =$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
Panel A: Mean absolute error												
QbSD approach												
QbSD-gSAV	0.251	0.208	0.178	0.319	0.259	0.217	0.326	0.236	0.185	0.459	0.319	0.243
QbSD-gAS	**0.123**	0.095	0.078	**0.178**	**0.135**	**0.109**	**0.216**	**0.142**	**0.106**	**0.335**	**0.216**	**0.157**
AL approach												
$AL_{Mult.}$-SAV	0.273	0.216	0.184	0.359	0.273	0.226	0.377	0.251	0.191	0.547	0.346	0.249
AL_{AR}-SAV	0.294	0.236	0.194	0.392	0.315	0.252	0.423	0.290	0.210	0.664	0.422	0.307
$AL_{Mult.}$-AS	0.166	0.113	0.086	0.252	0.168	0.131	0.313	0.169	0.115	0.522	0.280	0.187
AL_{AR}-AS	0.192	0.138	0.109	0.299	0.216	0.172	0.355	0.212	0.145	0.593	0.362	0.252
GAS	0.455	0.310	0.209	0.737	0.462	0.281	0.572	0.332	0.211	1.030	0.508	0.311
GARCH												
Normal	0.254	0.208	0.178	0.763	0.553	0.405	0.652	0.353	0.219	1.664	1.010	0.626
Student-t	0.251	0.210	0.180	0.522	0.427	0.346	0.319	0.241	0.193	1.051	0.732	0.524
Skew-t	0.252	0.211	0.181	0.254	0.206	0.172	0.321	0.243	0.194	0.350	0.245	0.184
GJR-GARCH												
Normal	0.245	0.199	0.169	0.764	0.553	0.405	0.659	0.356	0.217	1.664	1.010	0.626
Student-t	0.251	0.210	0.180	0.522	0.427	0.346	0.319	0.241	0.193	1.051	0.732	0.524
Skew-t	0.252	0.211	0.181	0.254	0.206	0.172	0.321	0.243	0.194	0.340	0.240	0.181
EGARCH	0.151	**0.095**	**0.068**	0.792	0.576	0.422	0.672	0.338	0.161	1.708	1.048	0.655
Panel B: Root mean squared error												
QbSD approach												
QbSD-gSAV	0.341	0.281	0.240	0.450	0.359	0.299	0.459	0.333	0.263	0.715	0.491	0.372
QbSD-gAS	**0.169**	**0.133**	0.110	**0.255**	**0.195**	**0.159**	**0.297**	**0.204**	**0.157**	**0.494**	**0.342**	**0.261**
AL approach												
$AL_{Mult.}$-SAV	0.373	0.292	0.248	0.517	0.385	0.317	0.557	0.363	0.276	0.883	0.705	0.384
AL_{AR}-SAV	0.394	0.321	0.257	0.568	0.458	0.357	0.613	0.408	0.294	1.157	0.705	0.521
$AL_{Mult.}$-AS	0.231	0.162	0.123	0.367	0.246	0.198	0.456	0.259	0.179	0.853	0.458	0.347
AL_{AR}-AS	0.278	0.194	0.154	0.455	0.330	0.258	0.563	0.348	0.211	1.021	0.675	0.405
GAS	0.586	0.415	0.272	1.056	0.632	0.389	1.029	0.464	0.287	1.973	0.844	0.475
GARCH												
Normal	0.337	0.278	0.239	0.874	0.644	0.481	0.772	0.456	0.310	1.967	1.197	0.747
Student-t	0.341	0.285	0.244	0.629	0.517	0.421	0.485	0.367	0.294	1.268	0.888	0.641
Skew-t	0.344	0.286	0.245	0.365	0.292	0.242	0.495	0.373	0.298	0.605	0.418	0.310
GJR-GARCH												
Normal	0.327	0.269	0.231	0.874	0.644	0.481	0.772	0.453	0.305	1.966	1.196	0.747
Student-t	0.341	0.285	0.244	0.629	0.517	0.421	0.485	0.367	0.294	1.269	0.889	0.641
Skew-t	0.343	0.286	0.245	0.365	0.292	0.242	0.496	0.374	0.298	0.572	0.404	0.302
EGARCH	0.195	0.133	**0.099**	0.904	0.666	0.496	0.756	0.403	0.218	2.146	1.348	0.872

Section C

This appendix reports additional rolling-window evaluations with $R = 250$ and $R = 2500$. Tables C1–C12 present 90% Model Confidence Set (MCS) rankings for VaR and joint VaR-ES at the 1%, 2.5%, and 5% levels.

Across windows, indices, and tails, the quantile-based scale-dynamics class performs very strongly overall. QbSD-gAS is the front-runner with $R = 250$: it ranks first in most VaR and joint VaR-ES exercises and is almost always in the top two. Its QAR extension (QAR-QbSD-gAS) typically follows closely-often finishing in the top three—but its long-window 1% VaR results slip on a few indices (see Table C7).

With the short window ($R = 250$), patterns are clear. For VaR-only (quantile scores; Tables C1–C3), QbSD-gAS is the *overall* top performer at 1%, 2.5%, and 5%. For joint VaR-ES (AL log scores; Tables C4–C6), skewed-t GARCH/GJR lead at 1% (Table C4) with QbSD-gAS a close second, while QbSD-gAS returns to the top at 2.5% and 5% (Tables C5–C6).

With the long window ($R = 2500$), leadership varies by tail and score. For VaR (Tables C7–C9), AL-based specifications dominate at 1% (Table C7), QbSD-gAS ranks first at 2.5% (Table C8), and at 5% EGARCH narrowly takes the lead, with QbSD-gAS essentially tied on average but second in the final ranking (Table C9); note that both have the same average rank (1.9), with QbSD-gAS recording more #1 finishes (4 of 8) while EGARCH is slightly more uniform across indices. For joint VaR-ES (Tables C10–C12), AL-based specifications lead at 1% and 2.5% (Tables C10–C11), whereas QbSD-gAS regains first place at 5% (Table C12).

Among benchmarks, skewed-t GARCH/GJR are the strongest non-QbSD competitors and are frequently retained in the MCS, especially for $R = 250$ and in VaR-only comparisons. Normal-based GARCH variants generally underperform on average, with two caveats: (i) EGARCH is a strong contender at 5% VaR for $R = 2500$ (Table C9), and (ii) GJR-GARCH-normal is competitive at $R = 250$ for 2.5% and 5% VaR (Tables C2–C3). Within the AL-based family, AS specifications consistently outperform their SAV counterparts. The gSAV variant of QbSD (QbSD-gSAV) typically trails QbSD-gAS, while QAR-QbSD-gSAV most often attains mid-table results.

Overall, these results align with the paper's main finding: QbSD-gAS is frequently the strongest—especially with the short window—and remains consistently competitive across indices and tail levels under the long window; QAR-QbSD-gAS is often close behind. At the extreme tail and with long windows, AL-based variants tend to lead, and EGARCH is a notable runner-up at 5% VaR.

Table C1. Ranking of 1% VaR forecasting models using the MCS procedure with quantile scores, rolling window $R = 250$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	4	2	1	2	1	1	4	8	2.1	1
QAR-QbSD-gAS	5	6	6	7	1	5	2	1	8	4.1	2
GARCH-skew-t	1	1	1	10	3	11	4	3	8	4.2	3
GJR-GARCH-skew-t	3	3	5	2	5	10	6	2	8	4.5	4
GJR-GARCH-t	10	2	9	4	8	3	7	8	8	6.4	5
AR-GARCH-skew-t	6	5	4	5	4	12	12	5	8	6.6	6
GARCH-t	4	9	8	8	11	2	5	9	8	7.0	7
AR-GJR-GARCH-skew-t	7	8	7	3	7	8	15	6	8	7.6	8
AR-GJR-GARCH-t	9	7	3	12	6	6	9	10	8	7.8	9
AR-GARCH-t	8	10	10	13	9	4	10	11	8	9.4	10
GJR-GARCH-normal	11	11	14	14	10	9	3	12	8	10.5	11
QAR-QbSD-gSAV	15		11	9	14	7	16	7	7	13.4	12
QbSD-gSAV	21	19	12	6	12	13	17	13	8	14.1	13
AR-GJR-GARCH-normal	18	18		17	16	17	11	14	7	17.4	14
GAS	13	16		19	17	20	14	15	7	17.8	15
AR-GAS	16	17	13	11		18	13		6	18.0	16
GARCH-normal	19	24		15	18	15	8		6	19.4	17
EGARCH	20	23		21	13	16	20		6	21.1	18
AR-EGARCH	23	27			15	14	19		5	22.8	19
AR-GARCH-normal		26		16		19	18		4	23.9	20
AR-AL$_{\text{Mult.}}$-AS	17	14		20					3	23.9	21
AR-AL$_{\text{AR}}$-AS		12		18					2	24.8	22
AR-AL$_{\text{Mult.}}$-SAV	14	20							2	25.2	23
AR-AL$_{\text{AR}}$-SAV	12	22							2	25.2	24
AL$_{\text{AR}}$-SAV		13							1	26.1	25
AL$_{\text{Mult.}}$-SAV		15							1	26.4	26
AL$_{\text{AR}}$-AS	22	21							2	26.4	27
AL$_{\text{Mult.}}$-AS		25							1	27.6	28

Table C2. Ranking of 2.5% VaR forecasting models using the MCS procedure with quantile scores, rolling window $R = 250$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	1	1	1	2	1	1	2	8	1.4	1
GARCH-skew-t	3	2	2	4	4	6	8	3	8	4.0	2
QAR-QbSD-gAS	1	5	18	3	1	4	2	1	8	4.4	3
GJR-GARCH-normal	5	4	5	11	3	2	3	7	8	5.0	4
GARCH-t	8	7	7	9	9	5	4	9	8	7.2	5
GJR-GARCH-t	11	6	9	10	7	3	5	8	8	7.4	6
GJR-GARCH-skew-t	4	3	3	5	6	8		4	7	7.6	7
AR-GARCH-skew-t	7	8	8	8	8	9		6	7	10.2	8
AR-GJR-GARCH-skew-t	6	10	6	7	12	10		5	7	10.5	9
GARCH-normal	13	9	10	12	10	11	6		7	12.4	10
AR-GJR-GARCH-normal	10	11	16	16		12	7	11	7	13.9	11
AR-GARCH-t	9	12	11	13	14	14			6	16.1	12
AR-GJR-GARCH-t	12	14	4	14	15	15			6	16.2	13
EGARCH	15	17	12	17	5	13			6	16.9	14
AR-EGARCH	14	20	14		11	7			5	18.8	15
QAR-QbSD-gSAV	18	18	15	2	13				5	18.8	16
QbSD-gSAV	25	13	13	6					4	21.1	17
AR-GARCH-normal	16	15		15		20			3	23.2	18
GAS	24		17	18					3	24.9	19
AR-GAS	27					10			2	25.6	20
AR-$AL_{Mult.}$-AS	22					16			2	25.8	21
$AL_{Mult.}$-AS		16							1	26.5	22
AL_{AR}-SAV	17								1	26.6	23
AL_{AR}-AS	26	19							2	26.6	24
AR-AL_{AR}-SAV	19								1	26.9	25
AR-$AL_{Mult.}$-SAV	20								1	27.0	26
$AL_{Mult.}$-SAV	21								1	27.1	27
AR-AL_{AR}-AS	23								1	27.4	28

Table C3. Ranking of 5% VaR forecasting models using the MCS procedure with quantile scores, rolling window $R = 250$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	3	1	2	1	1	1	1	3	8	1.6	1
QAR-QbSD-gAS	1	2	6	3	2	2		1	7	5.6	2
GARCH-skew-t	4	3	1	2	6	6		2	7	6.5	3
GJR-GARCH-normal	2	4	4	5	4	3		7	7	7.1	4
GJR-GARCH-skew-t	5	5	3	4	12	10		6	7	9.1	5
EGARCH	7	13	5	13	3	5		8	7	10.2	6
GARCH-t	12	8	9	7	11	7		12	7	11.8	7
AR-EGARCH	6	16	7	17	5	4		13	7	12.0	8
GJR-GARCH-t	13	7	8	9	15	8		10	7	12.2	9
AR-GJR-GARCH-skew-t	9	9	14	14	7	13		4	7	12.2	10
GARCH-normal	8	6	11	11	10	17		9	7	12.5	11
AR-GARCH-skew-t	11	10	12	16	8	11		5	7	12.6	12
AR-GJR-GARCH-normal	10	11	13	12	14	9		11	7	13.5	13
AR-GJR-GARCH-t	14	18	10	15	9	14			6	17.0	14
AR-GARCH-normal	16	12	15	10	16	15			6	17.5	15
AR-GARCH-t	15	15	17	18	13	12			6	18.2	16
QAR-QbSD-gSAV	17	19	16	8		16			5	20.0	17
QbSD-gSAV		20	18	6		18			4	21.8	18
AR-AL$_{\text{Mult.}}$-AS		14							1	26.2	19
AL$_{\text{Mult.}}$-AS		17							1	26.6	20
GAS	18								1	26.8	21
AL$_{\text{Mult.}}$-SAV							19		1	26.9	22
AR-AL$_{\text{AR}}$-AS		21							1	27.1	23
AR-GAS									0	28.0	24
AL$_{\text{AR}}$-SAV									0	28.0	25
AL$_{\text{AR}}$-AS									0	28.0	26
AR-AL$_{\text{Mult.}}$-SAV									0	28.0	27
AR-AL$_{\text{AR}}$-SAV									0	28.0	28

Table C4. Ranking of 1% VaR and ES forecasting models based on the MCS procedure using AL log scores, rolling window $R = 250$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
GJR-GARCH-skew-t	2	4	2	1	13	11	3	2	8	4.8	1
QbSD-gAS	9	2	6	5	6	1	1	10	8	5.0	2
GARCH-skew-t	1	1	1	3	9	13	4	9	8	5.1	3
QAR-QbSD-gAS	8	7	7	16	1	5	2	1	8	5.9	4
AR-AL$_{\text{Mult.}}$-AS	6	3	5	9	8	7	9	3	8	6.2	5
AL$_{\text{Mult.}}$-AS	10	8	3	10	10	12	10	4	8	8.4	6
AL$_{\text{Mult.}}$-SAV	5	6		12	3	4	7	5	7	8.8	7
AR-AL$_{\text{Mult.}}$-SAV	3	5	18	8	5	3	6		7	9.5	8
AR-GJR-GARCH-skew-t	11	11	9	2	14	10	17	7	8	10.1	9
AR-AL$_{\text{AR}}$-AS	4		4	7	4	6	5		6	10.8	10
AR-GARCH-skew-t	14	12	8	14	11	14	20	6	8	12.4	11
QAR-QbSD-gSAV	19	17	12	13	16	9	11	8	8	13.1	12
AL$_{\text{AR}}$-AS	7		11	19	12	8	8		6	15.1	13
AR-AL$_{\text{AR}}$-SAV	13	10		6	7	2			5	15.2	14
QbSD-gSAV	20	18	14	4	15	17	12		7	16.0	15
GJR-GARCH-t	17	13	15	11	18	15	14		7	16.4	16
GARCH-t	15	15	16	15	19	16	13		7	17.1	17
AR-GJR-GARCH-t	18	14	10	17	17	19	19		7	17.8	18
AR-GARCH-t	16	16	13	18	20	18	22		7	18.9	19
AL$_{\text{AR}}$-SAV	12	9			2		21		4	19.5	20
AR-GAS	21		19			21	15		3	24.4	21
GJR-GARCH-normal					21	20	16		3	24.6	22
GAS	22		17				18		3	24.6	23
AR-GJR-GARCH-normal							23		1	27.4	24
GARCH-normal									0	28.0	25
EGARCH									0	28.0	26
AR-GARCH-normal									0	28.0	27
AR-EGARCH									0	28.0	28

Table C5. Ranking of 2.5% VaR and ES forecasting models based on the MCS procedure using AL log scores, rolling window $R = 250$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	1	2	2	7	1	1	7	8	2.9	1
QAR-QbSD-gAS	1	6	7	11	1	2	5	1	8	4.2	2
GARCH-skew-t	3	2	1	6		7	9	4	7	7.5	3
GJR-GARCH-skew-t	4	3	3	4		9	10	3	7	8.0	4
AR-$\text{AL}_{\text{Mult.}}$-AS	6		5	9	6	5	7	2	7	8.5	5
$\text{AL}_{\text{Mult.}}$-AS	8	7	6	7	3	4	8		7	8.9	6
QAR-QbSD-gSAV	12	11	12	1		13	14	8	7	12.4	7
AR-GJR-GARCH-skew-t	9	8	8	5		12		6	6	13.0	8
AR-GARCH-skew-t	11	9	10	10		11		5	6	14.0	9
AL_{AR}-AS	13	12			5	6	2		5	15.2	10
AR-AL_{AR}-AS	21		4	8		3	6		5	15.8	11
QbSD-gSAV	16	13	16	3		14	11		6	16.1	12
GJR-GARCH-t	18	10	14	13		8	13		6	16.5	13
GARCH-t	15	15	13	12		10	12		6	16.6	14
AR-$\text{AL}_{\text{Mult.}}$-SAV	5	4	9		4				4	16.8	15
$\text{AL}_{\text{Mult.}}$-SAV	7	5			8		4		4	17.0	16
AL_{AR}-SAV	14				2		3		3	19.9	17
AR-GJR-GARCH-t	19		11	14		16			4	21.5	18
AR-GARCH-t	17		17	15		17			4	22.2	19
GJR-GARCH-normal	20		15			15			3	23.8	20
AR-AL_{AR}-SAV	10	14							2	24.0	21
GAS			18						1	26.8	22
AR-GJR-GARCH-normal		19							1	26.9	23
GARCH-normal		20							1	27.0	24
EGARCH									0	28.0	25
AR-GARCH-normal									0	28.0	26
AR-EGARCH									0	28.0	27
AR-GAS									0	28.0	28

Table C6. Ranking of 5% VaR and ES forecasting models based on the MCS procedure using AL log scores, rolling window $R = 250$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	1	1	1	2	1	1	10	8	2.4	1
QAR-QbSD-gAS	1	4	6	6	1	5	8	1	8	4.0	2
AL$_{\text{Mult.}}$-AS	5	5	5	8	5	9	5	4	8	5.8	3
AR-AL$_{\text{Mult.}}$-AS	4		4	7	6	8	7	3	7	8.4	4
GARCH-skew-t	6	3	2	3				6	5	13.0	5
GJR-GARCH-skew-t	7	6	3	2				7	5	13.6	6
AL$_{\text{Mult.}}$-SAV	3	2			4	6			4	15.9	7
AR-GARCH-skew-t	8	9	9	10				9	5	16.1	8
AR-AL$_{\text{AR}}$-AS			7		4	6	2	4	4	16.4	9
AR-GJR-GARCH-skew-t	10		8	9				8	4	18.4	10
AL$_{\text{AR}}$-AS					3	4	5	3	4	19.0	11
AR-AL$_{\text{Mult.}}$-SAV					3	7	3		3	19.1	12
GARCH-t	9	10	14	11					4	19.5	13
GJR-GARCH-t	16	7	12	12					4	19.9	14
GJR-GARCH-normal	13	8	13	14					4	20.0	15
QAR-QbSD-gSAV	15		10	4					3	21.1	16
AR-AL$_{\text{AR}}$-SAV					2	2			2	21.5	17
AR-GJR-GARCH-t	12		11	15					3	22.2	18
QbSD-gSAV				5					1	25.1	19
AR-GARCH-t	11								1	25.9	20
GARCH-normal			13						1	26.1	21
AR-GJR-GARCH-normal	14								1	26.2	22
AR-GARCH-normal			16						1	26.5	23
EGARCH									0	28.0	24
AR-EGARCH									0	28.0	25
GAS									0	28.0	26
AR-GAS									0	28.0	27
AL$_{\text{AR}}$-SAV									0	28.0	28

Table C7. Ranking of 1% VaR forecasting models using the MCS procedure with quantile scores, rolling window $R = 2500$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
AR-AL$_{\text{AR}}$-AS	1	1	1	9	3	4	6	5	8	3.8	1
AR-AL$_{\text{Mult.}}$-AS	7	9	6	14	6	6	4	2	8	6.8	2
AL$_{\text{Mult.}}$-AS	18	2	7	17	4	2	3	3	8	7.0	3
AL$_{\text{AR}}$-AS	4	8	9	13	2	20	2	4	8	7.8	4
QbSD-gAS		6	25	1	1	1	1	6	7	8.6	5
AR-AL$_{\text{Mult.}}$-SAV	2	3	3	16	16	18	7	8	8	9.1	6
GJR-GARCH-skew-t	17	7	11	3	8	8	14	16	8	10.5	7
AL$_{\text{Mult.}}$-SAV	5	12	4	6	15	21	10	11	8	10.5	8
AR-AL$_{\text{AR}}$-SAV	3	4	2	25	11	17	13	12	8	10.9	9
AR-GJR-GARCH-skew-t	8	5	15	8	13	10	18	13	8	11.2	10
GARCH-skew-t	13	17	13	7	10	5	15	14	8	11.8	11
AR-GARCH-skew-t	9	10	10	10	14	9	17	15	8	11.8	12
AL$_{\text{AR}}$-SAV	6	11	5	23	12	19	16	10	8	12.8	13
QAR-QbSD-gAS	26	13		22	5	3	5	1	7	12.9	14
GARCH-t	11	14	19	2	17	11	8	22	8	13.0	15
EGARCH	19	20	14	5	21	13	20	7	8	14.9	16
AR-GJR-GARCH-t	10	19	16	24	18	7	12	24	8	16.2	17
AR-GARCH-t	16	18	17	15	20	15	11	23	8	16.9	18
GJR-GARCH-t	12	15	20	26	19	14	9	21	8	17.0	19
QAR-QbSD-gSAV	25	16	21	11	9	16	23	17	8	17.2	20
QbSD-gSAV		23	24	4	7	26	22	19	7	19.1	21
AR-EGARCH	21	22	18	12	22	27	27	9	8	19.8	22
GAS	14	25	12		27	12	26		6	21.5	23
GJR-GARCH-normal	22	24	22	19	25	22	19	20	8	21.6	24
AR-GAS	15	21	8		23	28	25		6	22.0	25
GARCH-normal	20	26	23	18	24	23	21		7	22.9	26
AR-GJR-GARCH-normal	24	28	26	21	28	24	28	18	8	24.6	27
AR-GARCH-normal	23	27	27	20	26	25	24		7	25.0	28

Table C8. Ranking of 2.5% VaR forecasting models using the MCS procedure with quantile scores, rolling window $R = 2500$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	1	2	1	2	1	2	2	2	8	1.6	1
$AL_{Mult.}$-AS	3	4	3	6	7	5	1	1	8	3.8	2
AR-$AL_{Mult.}$-AS	4	1	7	4	4	4	6	6	8	4.5	3
AL_{AR}-AS	2	3	2	15	5	3	4	3	8	4.6	4
EGARCH	6	5	10	1	6	6	3	7	8	5.5	5
AR-AL_{AR}-AS	5	22	4	5	3	1	5	4	8	6.1	6
QAR-QbSD-gAS	7	11	24	7	2	7		5	7	11.4	7
GJR-GARCH-normal	17	6	15	21	14	14	7	10	8	13.0	8
AR-EGARCH	8	15	19	3	8	23		8	7	14.0	9
$AL_{Mult.}$-SAV	9	7	9	14	17	25	9		7	14.8	10
GARCH-skew-t	10	16	11	10	11	8			6	15.2	11
GARCH-normal	13	12	18	20	16	15	8		7	16.2	12
GJR-GARCH-skew-t	15	14	12	16	10	9			6	16.5	13
GARCH-t	12	9	13	13	15	16			6	16.8	14
AL_{AR}-SAV	18	8	8	12	22	26			6	18.8	15
GJR-GARCH-t	14	10	14	18	21	18			6	18.9	16
AR-GJR-GARCH-skew-t	11	17	21	17	20	12			6	19.2	17
AR-GARCH-skew-t	23	20	20	11	19	11			6	20.0	18
GAS		13		27	9	22		9	5	20.5	19
AR-$AL_{Mult.}$-SAV	16	18	5	19	27	24			6	20.6	20
QbSD-gSAV	26	26		9	12	10			5	20.9	21
QAR-QbSD-gSAV	25	23	25	8	13	17			6	20.9	22
AR-GJR-GARCH-t	19	24	17	24	18	13			6	21.4	23
AR-AL_{AR}-SAV	22	19	6	22	25	27			6	22.1	24
AR-GJR-GARCH-normal	20	28	22	26	24	19		12	7	22.4	25
AR-GARCH-t	24	25	16	23	23	21			6	23.5	26
AR-GARCH-normal	21	27	23	25	26	20			6	24.8	27
AR-GAS	27	21				28		11	4	24.9	28

Table C9. Ranking of 5% VaR forecasting models using the MCS procedure with quantile scores, rolling window $R = 2500$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
EGARCH	2	2	2	2	2	1	2	2	8	1.9	1
QbSD-gAS	1	1	1	3	1	2	3	3	8	1.9	2
$AL_{Mult.}$-AS	4	4	3	5	5	3	4	6	8	4.2	3
AL_{AR}-AS	6	3	15	6	3	5	1	5	8	5.5	4
QAR-QbSD-gAS	7	7	6	4	6	8	11	1	8	6.2	5
AR-EGARCH	8	6	4	1	11	6	7	8	8	6.4	6
AR-AL_{AR}-AS	3	5	26	8	4	4	5	4	8	7.4	7
AR-$AL_{Mult.}$-AS	5	8	5	7	21	7	6	7	8	8.2	8
GJR-GARCH-normal	16	12	11		10	9	14	9	7	13.6	9
GARCH-normal	15	14	13		12	10	15		6	16.9	10
GARCH-t	18	18	14		8	15	13		6	17.8	11
GJR-GARCH-t	20	19	12		7	16	12		6	17.8	12
GJR-GARCH-skew-t	10	9	7		13	20			5	17.9	13
AR-GJR-GARCH-normal	9	21	17		25	12	21	11	7	18.0	14
GARCH-skew-t	12	10	9		14	22	22		6	18.1	15
QbSD-gSAV	22	16	20	13	15		8		6	18.8	16
AR-GJR-GARCH-t	17	26	24		9	14	18		6	20.5	17
GAS			28	9	28	17	16	10	6	20.5	18
AL_{AR}-SAV	21	17	16	12	23		17		6	20.2	19
AR-GARCH-normal	13	20	19		22	13	20		6	20.4	20
AR-GAS		13	27		17	11	9		5	20.1	21
AR-GJR-GARCH-skew-t	11	11	10		16	19			5	18.9	22
$AL_{Mult.}$-SAV	26	24	21	10	20		10		6	20.9	23
AR-GARCH-t	19	27	22		18	18	19		6	22.4	24
AR-$AL_{Mult.}$-SAV	24	23	18	11	26		23		6	22.6	25
QAR-QbSD-gSAV	23	22	23		20		22		5	24.9	26
AR-AL_{AR}-SAV	25	25	25		27				4	26.8	27

Table C10. Ranking of 1% VaR and ES forecasting models based on the MCS procedure using AL log scores, rolling window $R = 2500$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
AR-AL$_{\text{AR}}$-AS	1	6	1	10	1	4	3	5	8	3.9	1
AR-AL$_{\text{Mult.}}$-AS	6	1	6	12	4	5	2	8	8	5.5	2
QbSD-gAS	15	3	14	4	2	3	4	1	8	5.8	3
AL$_{\text{Mult.}}$-AS	12	2	11	14	3	1	6	4	8	6.6	4
GJR-GARCH-skew-t	9	7	7	2	5	7	7	16	8	7.5	5
AR-AL$_{\text{Mult.}}$-SAV	2	5	3	15	12	16	9	7	8	8.6	6
QAR-QbSD-gAS	11	4		11	11	2	5	3	7	9.4	7
GARCH-skew-t	7	20	8	7	7	6	8	15	8	9.8	8
AL$_{\text{Mult.}}$-SAV	4	12	4	9	13	18	12	6	8	9.8	9
AL$_{\text{AR}}$-AS	13	9		13	6	14	1	2	7	10.8	10
AR-GJR-GARCH-skew-t	5	8	12	5	15	10	18	18	8	11.4	11
AR-AL$_{\text{AR}}$-SAV	3	19	2	18	10	20	10	10	8	11.5	12
AR-GARCH-skew-t	10	11	10	8	16	9	16	17	8	12.1	13
AL$_{\text{AR}}$-SAV	8	13	5	20	14	19	11	9	8	12.4	14
QAR-QbSD-gSAV	21	10	13	6	9	15	15	13	8	12.8	15
GARCH-t	16	14	15	3	18	11	13	20	8	13.8	16
QbSD-gSAV	23	21	17	1	8	17	17	11	8	14.4	17
GJR-GARCH-t	17	15	20	22	19	12	14	19	8	17.2	18
AR-GJR-GARCH-t	19	16	18	21	17	8	20	21	8	17.5	19
AR-GARCH-t	20	18	19	16	20	13	19	22	8	18.4	20
AR-GAS	14	17	9		21		22		5	20.9	21
GAS	18	24	16		22	21	21		6	22.2	22
EGARCH	22	22		17		22		12	5	22.4	23
AR-EGARCH		23		19		23		14	4	23.9	24
GJR-GARCH-normal				24		24		23	3	26.4	25
GARCH-normal				23		25			2	27.0	26
AR-GARCH-normal				25					1	27.6	27
AR-GJR-GARCH-normal								26	1	27.8	28

Table C11. Ranking of 2.5% VaR and ES forecasting models based on the MCS procedure using AL log scores, rolling window $R = 2500$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
AR-AL$_{\text{Mult.}}$-AS	1	2	6	1	1	1	4	5	8	2.6	1
AL$_{\text{Mult.}}$-AS	6	3	1	3	6	4	2	2	8	3.4	2
QbSD-gAS	5	1	8	2	3	5	5	1	8	3.8	3
QAR-QbSD-gAS	3	4	12	5	5	6	6	3	8	5.5	4
AL$_{\text{AR}}$-AS	2	5	4	21	4	2	3	6	8	5.9	5
AR-AL$_{\text{AR}}$-AS	4	20	9	4	2	3	1	4	8	5.9	6
GJR-GARCH-skew-t	10	6	11	7	9	8	12		7	11.4	7
AL$_{\text{Mult.}}$-SAV	7	7	5	6	13	18	8		7	11.5	8
AR-AL$_{\text{Mult.}}$-SAV	8	10	3	10	17	16	9		7	12.6	9
GARCH-skew-t	9	19	10	9	10	7	11		7	12.9	10
AL$_{\text{AR}}$-SAV	13	12	7	17	8	21	7		7	14.1	11
AR-GJR-GARCH-skew-t	11	8	16	14	15	11	14		7	14.6	12
AR-GARCH-skew-t	18	11	15	12	16	9	15		7	15.5	13
AR-AL$_{\text{AR}}$-SAV	12	26	2	18	7	22	10		7	15.6	14
EGARCH	19	9		11	23	19	13	7	7	16.1	15
QAR-QbSD-gSAV	14	16	17	8	12	10			6	16.6	16
GARCH-t	15	14	13	16	18	15	16		7	16.9	17
GJR-GARCH-t	16	15	14	19	21	17			6	19.8	18
QbSD-gSAV	23	22		13	14	12	18		6	19.8	19
AR-GJR-GARCH-t	17	17	19	22	19	13			6	20.4	20
AR-EGARCH	20	13		15	24	23	17		6	21.0	21
AR-GARCH-t	21	18	18	20	22	20			6	21.9	22
GAS		23	20	27	11	14			5	22.4	23
AR-GAS	22	24	21		20	26			5	24.6	24
GJR-GARCH-normal		21		24		24		23	3	26.1	25
GARCH-normal		25		23		25			3	26.6	26
AR-GARCH-normal				25					1	27.6	27
AR-GJR-GARCH-normal				26		27			2	27.6	28

Table C12. Ranking of 5% VaR and ES forecasting models based on the MCS procedure using AL log scores, rolling window $R = 2500$

	S&P	DJIA	NASDAQ	STOXX	FTSE	DAX	CAC	TSX	#	Avg. rank	Final rank
QbSD-gAS	2	2	2	1	1	5	3	2	8	2.2	1
$\text{AL}_{\text{Mult.}}$-AS	3	1	1	4	4	1	4	4	8	2.8	2
AL_{AR}-AS	5	8	14	8	3	3	1	3	8	5.6	3
QAR-QbSD-gAS	4	3	20	2	6	6	5	1	8	5.9	4
AR-AL_{AR}-AS	11	7	5	20	2	4	2	6	8	7.1	5
AR-$\text{AL}_{\text{Mult.}}$-AS	1	6	8	5	5	2		5	7	7.5	6
EGARCH	9	4	22	6	8	7		7	7	11.4	7
AR-EGARCH	12	5	25	3	11	8		8	7	12.5	8
GJR-GARCH-skew-t	7	9	3	12	16	13			6	14.5	9
GARCH-skew-t	8	11	4	9	17	15			6	15.0	10
AR-GJR-GARCH-skew-t	6	10	10	16	18	11			6	15.9	11
GARCH-t	10	13	16	10	15	12			6	16.5	12
QAR-QbSD-gSAV	15	12	11	7	12	22			6	16.9	13
AR-GARCH-skew-t	17	14	7	14	19	16			6	17.9	14
GJR-GARCH-t	13	15	17	18	13	14			6	18.2	15
AR-GJR-GARCH-t	14	18	24	21	14	10			6	19.6	16
AR-$\text{AL}_{\text{Mult.}}$-SAV	18	20	6	13	24	24			6	20.1	17
AL_{AR}-SAV	20	17	12	22	9				5	20.5	18
AR-AL_{AR}-SAV	21	23	9	19	10				5	20.8	19
AR-GARCH-t	16	19	23	17	21	17			6	21.1	20
$\text{AL}_{\text{Mult.}}$-SAV	19	16	13	15	22				5	21.1	21
QbSD-gSAV	22	21	21	11	20	23			6	21.8	22
GAS			15		23	9			3	23.4	23
GJR-GARCH-normal	25	22	19	23	25	18			6	23.5	24
AR-GAS		25	18		7				3	23.8	25
GARCH-normal	23	24		24		19			4	25.2	26
AR-GJR-GARCH-normal	24	26		26		20			4	26.0	27
AR-GARCH-normal	26	27		25		21			4	26.4	28